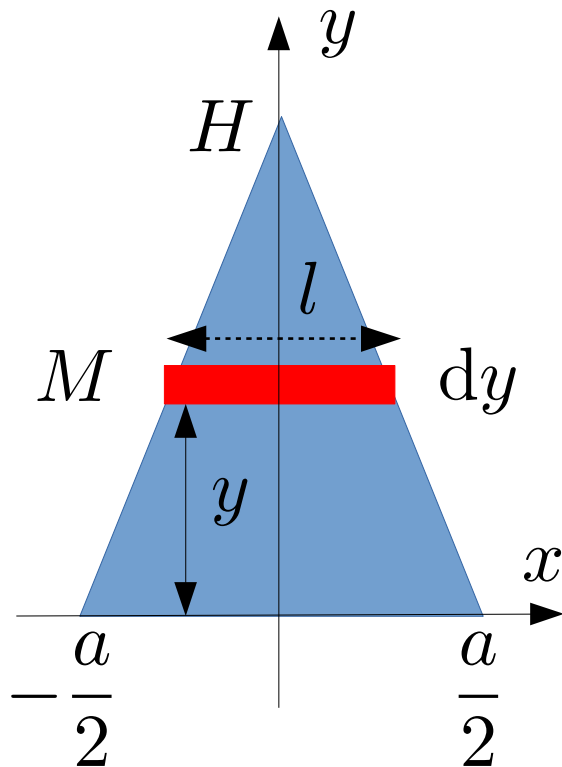


# جلسه ششم

## مکانیک تحلیلی

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اسفند ۹۸

# حرکت اجسام صلب در صفحه



$$\begin{aligned} \mathbb{I}_x &= \int y^2 dm = \frac{2M}{aH} \int y^2 dS && \text{مثلت} \\ &= \frac{2M}{aH} \int y^2 l dy = \frac{2M}{aH} \int y^2 \frac{a}{H} (H - y) dy \\ &= \frac{2M}{H^2} \int_0^H y^2 (H - y) dy = \frac{1}{6} M H^2 \end{aligned}$$

$$\mathbb{I}_x = \frac{1}{6} M H^2$$

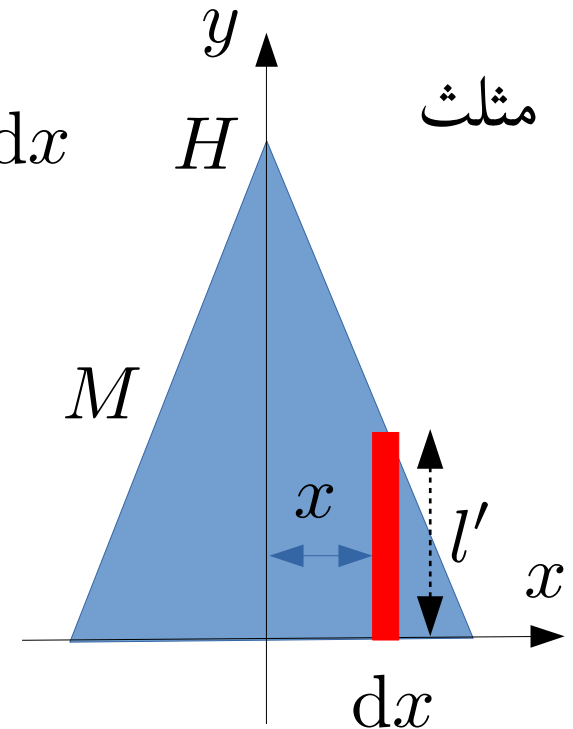
# حرکت اجسام صلب در صفحه

$$I_y = \int x^2 dm = \frac{2M}{aH} \int x^2 dS = \frac{2M}{aH} \int x^2 l' dx$$

$$l'(x) = \begin{cases} \frac{H}{a}(a + 2x), & -a/2 \leq x \leq 0 \\ \frac{H}{a}(a - 2x), & 0 \leq x \leq a/2 \end{cases}$$

$$= \frac{2M}{a^2} \left[ \int_{-a/2}^0 x^2(a + 2x) dx \right.$$

$$\left. + \int_0^{a/2} x^2(a - 2x) dx \right] = \frac{1}{24} Ma^2 \quad I_y = \frac{1}{24} Ma^2$$



# حرکت اجسام صلب در صفحه

قضیه محورهای عمودی

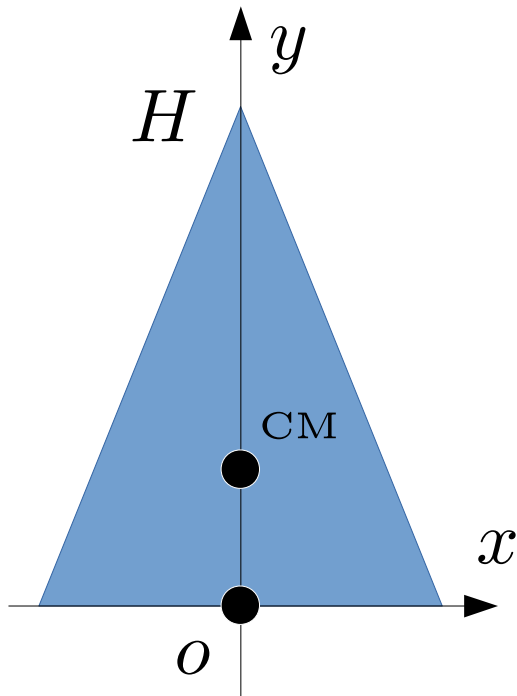
$$\mathbb{I}_o = \mathbb{I}_x + \mathbb{I}_y$$

$$\mathbb{I}_o = \frac{1}{24} M (a^2 + 4H^2)$$

قضیه محورهای موازی

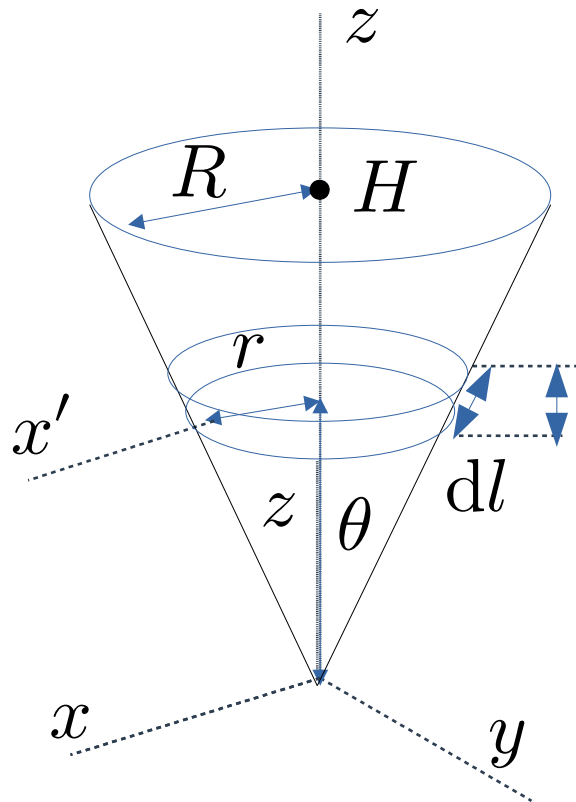
$$\mathbb{I}_o = \mathbb{I}_{CM} + M \left( \frac{H}{3} \right)^2$$

$$\mathbb{I}_{CM} = \mathbb{I}_o - M \left( \frac{H}{3} \right)^2 = \frac{1}{24} M a^2 + \frac{1}{18} M H^2$$



# حرکت اجسام صلب در صفحه

مخروط توخالی

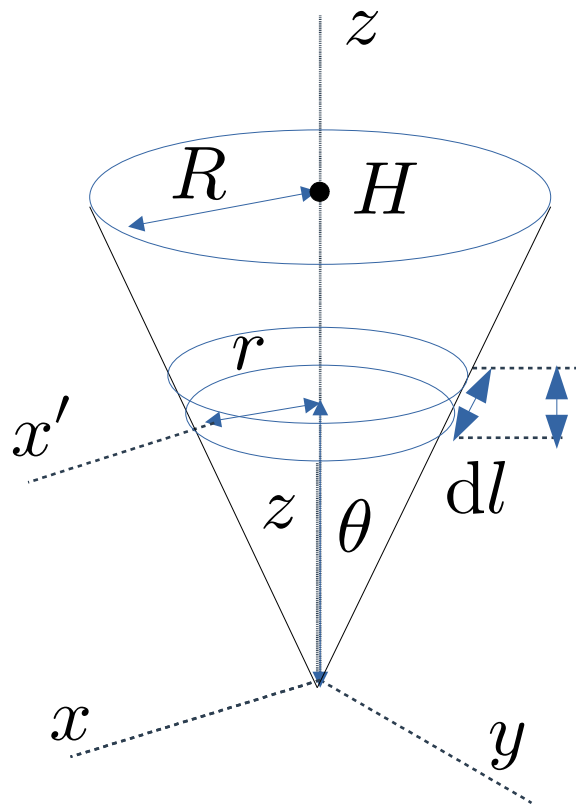


$$dl = \frac{dz}{\cos \theta} = \frac{\sqrt{R^2 + H^2}}{H} dz$$

$$dS = 2\pi r dl = 2\pi \frac{R}{H} z \frac{\sqrt{R^2 + H^2}}{H} dz$$

$$S = \int dS = 2\pi \frac{R\sqrt{R^2 + H^2}}{H^2} \int_0^H z dz = \pi R \sqrt{R^2 + H^2}$$

# حرکت اجسام صلب در صفحه



مخروط توخالی

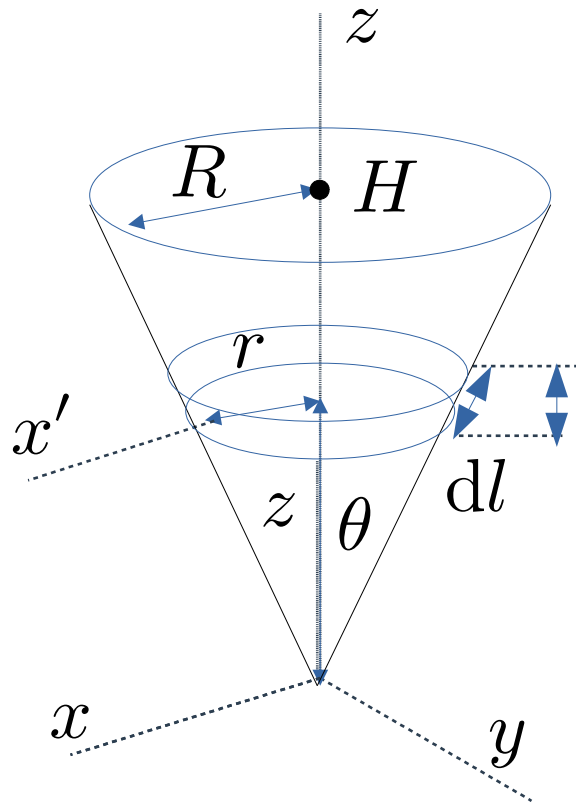
$$dI_z = r^2 dm$$

$$dm = \frac{2M}{H^2} z dz, \quad r = \frac{R}{H} z$$

$$dI_z = \frac{R^2}{H^2} z^2 \frac{2M}{H^2} z dz = \frac{2MR^2}{H^4} z^3 dz$$

$$I_z = \frac{1}{2} MR^2$$

# حرکت اجسام صلب در صفحه



مخروط توخالی

$$dI_x = dI_{x'} + z^2 dm$$

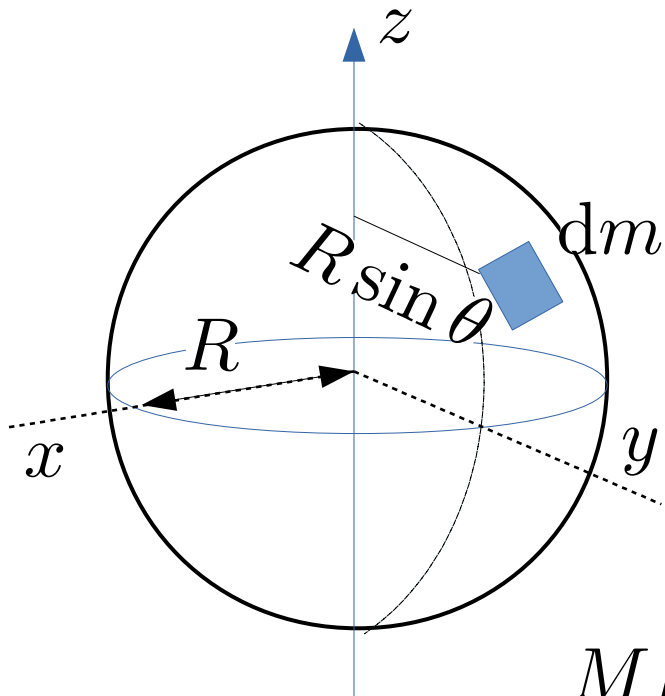
$$dI_{x'} = \frac{1}{2} r^2 dm$$

$$dm = \frac{2M}{H^2} z dz, \quad r = \frac{R}{H} z$$

$$dI_x = \left[ \frac{R^2}{2H^2} + 1 \right] z^2 \frac{2M}{H^2} z dz$$

$$I_x = \left[ \frac{R^2}{2H^2} + 1 \right] \frac{2M}{H^2} \int_0^H z^3 dz = \frac{1}{4} M R^2 + \frac{1}{2} M H^2$$

# حرکت اجسام صلب در صفحه



$$I_z = \int r^2 dm$$

گروه تو خالی

$$dm = \frac{M}{4\pi} \sin \theta d\theta d\phi$$

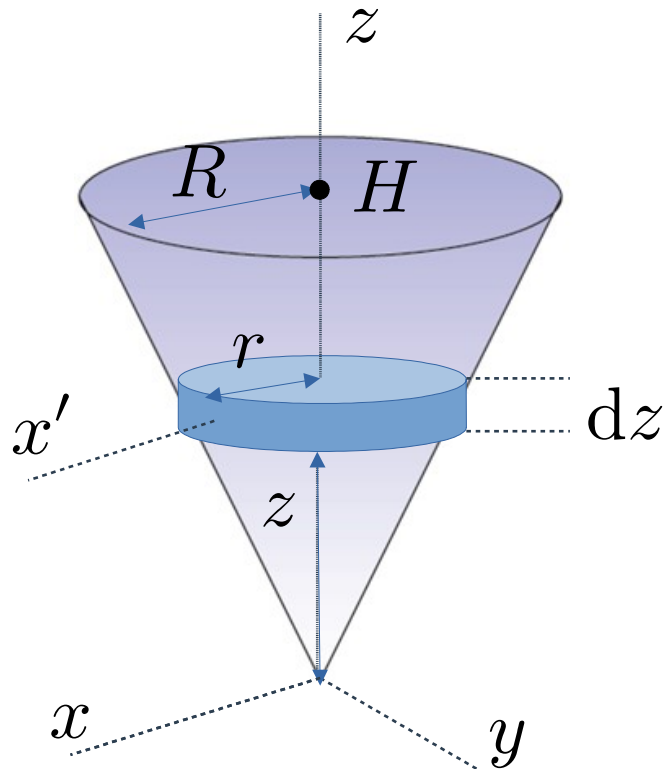
$$I_z = \int (R \sin \theta)^2 \frac{M}{4\pi} \sin \theta d\theta d\phi$$

$$I_z = \frac{MR^2}{4\pi} \left( \int_0^\pi \sin^3 \theta d\theta \right) \left( \int_0^{2\pi} d\phi \right)$$

$$I_z = \frac{MR^2}{2\pi} \left( -\cos \theta + \frac{1}{3} \cos^3 \theta \right)_0^\pi 2\pi = \frac{2}{3} MR^2$$



# حرکت اجسام صلب در صفحه



مخروط توپر

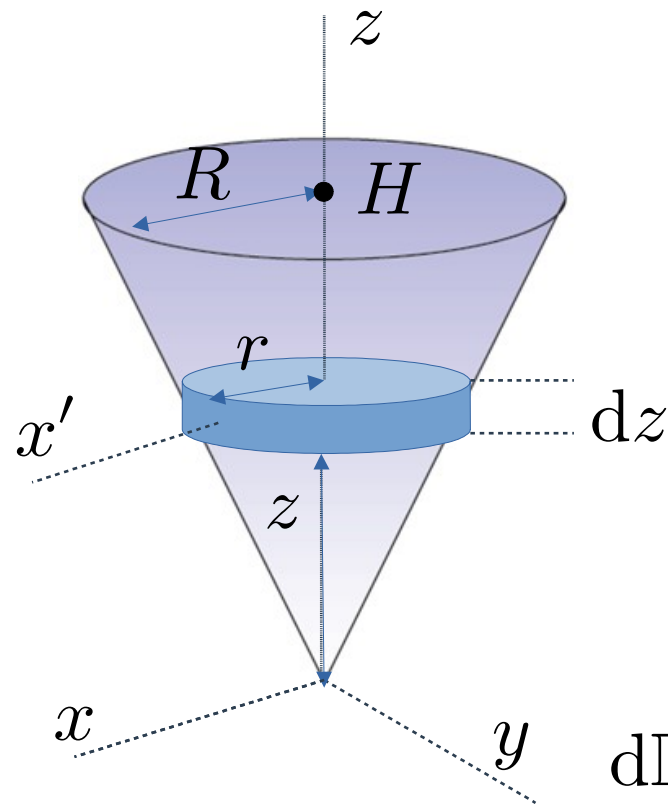
$$dI_z = \frac{1}{2} r^2 dm$$

$$r = \frac{R}{H} z, \quad dm = \frac{3M}{H^3} z^2 dz$$

$$dI_z = \frac{1}{2} \left( \frac{R^2}{H^2} z^2 \right) \frac{3M}{H^3} z^2 dz$$

$$I_z = \frac{3MR^2}{2H^5} \int_0^H z^4 dz = \frac{3}{10} MR^2$$

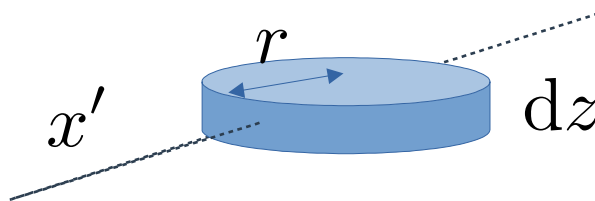
# حرکت اجسام صلب در صفحه



مخروط توپر

$$dI_x = dI_{x'} + z^2 dm$$

$$dI_{x'} = \frac{1}{4} r^2 dm$$

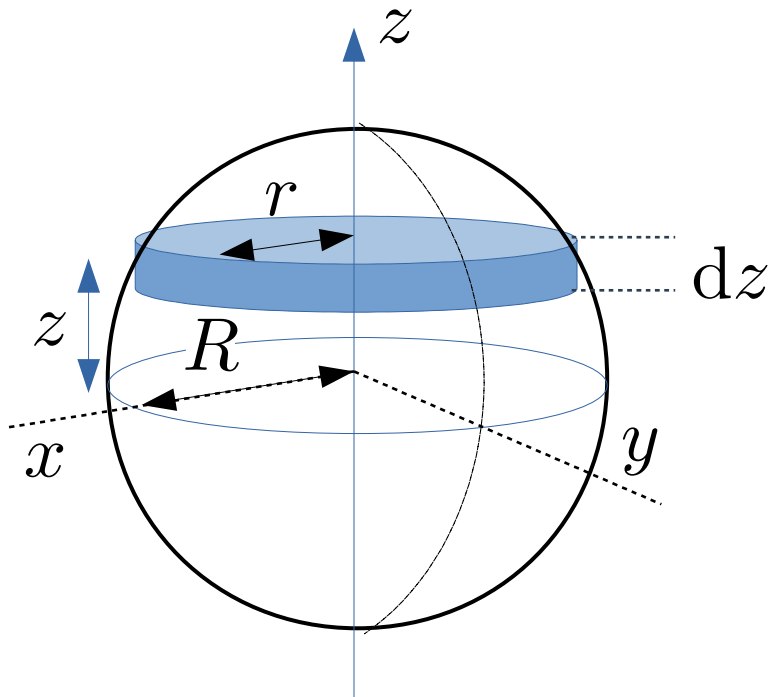


$$r = \frac{R}{H} z, \quad dm = \frac{3M}{H^3} z^2 dz$$

$$dI_x = \frac{1}{4} \left( \frac{R^2}{H^2} z^2 \right) \frac{3M}{H^3} z^2 dz + z^2 \frac{3M}{H^3} z^2 dz$$

$$I_x = 3M \left( \frac{R^2}{4H^5} + \frac{1}{H^3} \right) \int_0^H z^4 dz \Rightarrow I_x = \frac{3MR^2}{20} + \frac{3MH^2}{5}$$

# حرکت اجسام صلب در صفحه



گُره توپیر

$$d\mathbb{I}_z = \frac{1}{2} r^2 dm$$

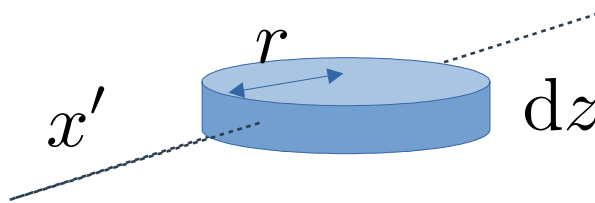
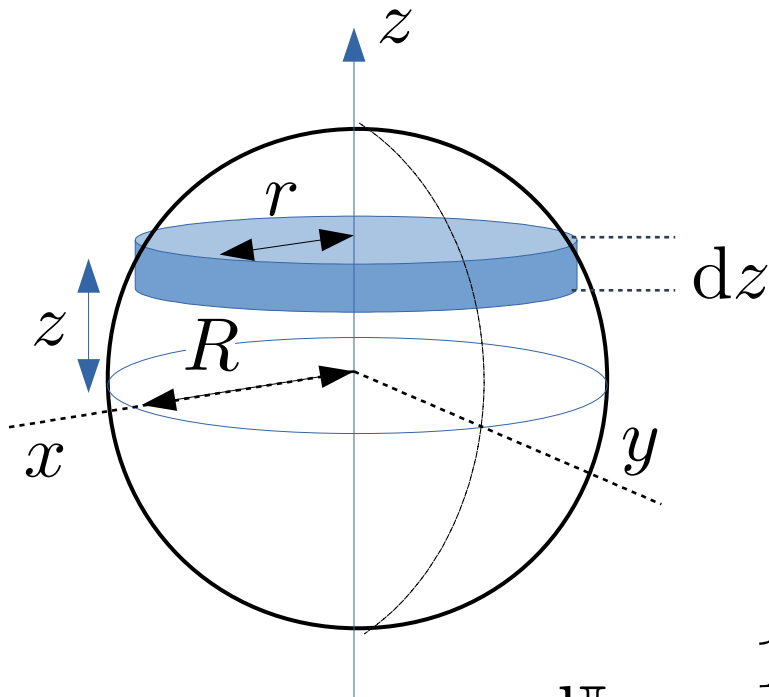
$$r = \sqrt{R^2 - z^2}$$

$$dm = \frac{3M}{4R^3} (R^2 - z^2) dz$$

$$d\mathbb{I}_z = \frac{3M}{8R^3} (R^2 - z^2)^2 dz$$

$$\mathbb{I}_z = \frac{3M}{8R^3} \int_{-R}^R (R^2 - z^2)^2 dz = \frac{2}{5} MR^2$$

# حرکت اجسام صلب در صفحه



گُره توپیر

$$d\mathbb{I}_x = d\mathbb{I}_{x'} + z^2 dm$$

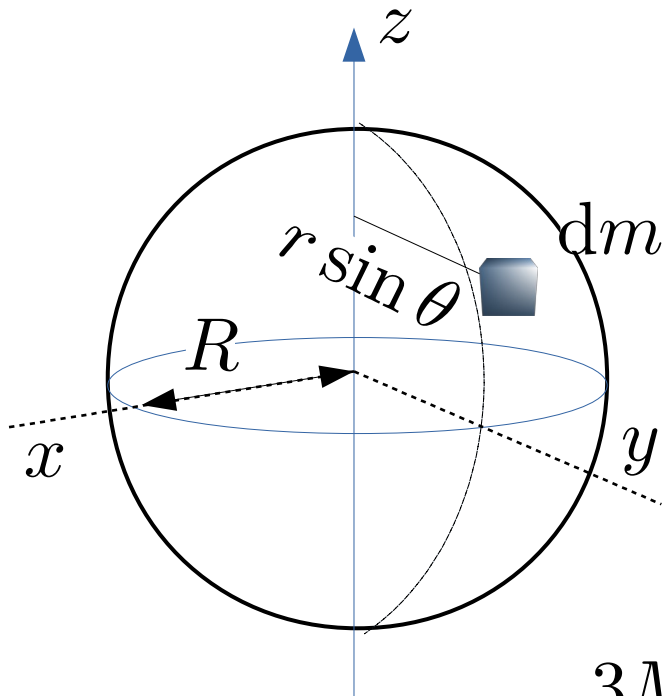
$$d\mathbb{I}_{x'} = \frac{1}{4} r^2 dm$$

$$r = \sqrt{R^2 - z^2}, \quad dm = \frac{3M}{4R^3} (R^2 - z^2) dz$$

$$d\mathbb{I}_x = \frac{1}{4} \frac{3M}{4R^3} (R^2 - z^2)^2 dz + \frac{3M}{4R^3} z^2 (R^2 - z^2) dz$$

$$\mathbb{I}_x = \frac{1}{4} \frac{3M}{4R^3} \left( \int_{-R}^R [(R^2 - z^2)^2 + 4z^2(R^2 - z^2)] dz \right) = \frac{2}{5} MR^2$$

# حرکت اجسام صلب در صفحه



$$\mathbb{I}_z = \int r^2 dm \quad \text{گره توپیر}$$

$$dm = \frac{3M}{4\pi R^3} r^2 \sin \theta dr d\theta d\phi$$

$$\mathbb{I}_z = \int (r \sin \theta)^2 \frac{3M}{4\pi R^3} r^2 \sin \theta dr d\theta d\phi$$

$$\mathbb{I}_z = \frac{3M}{4\pi R^3} \left( \int_0^R r^4 dr \right) \left( \int_0^\pi \sin^3 \theta d\theta \right) \left( \int_0^{2\pi} d\phi \right)$$

$$\mathbb{I}_z = \frac{3M}{4\pi R^3} \frac{1}{5} R^5 \left( -\cos \theta + \frac{1}{3} \cos^3 \theta \right)_0^\pi 2\pi = \frac{2}{5} M R^2$$