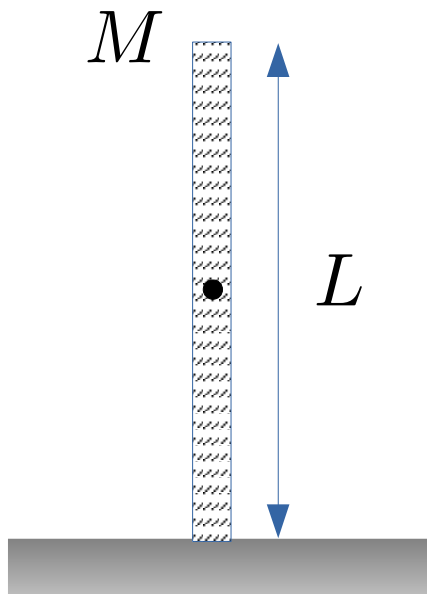


جلسه نهم

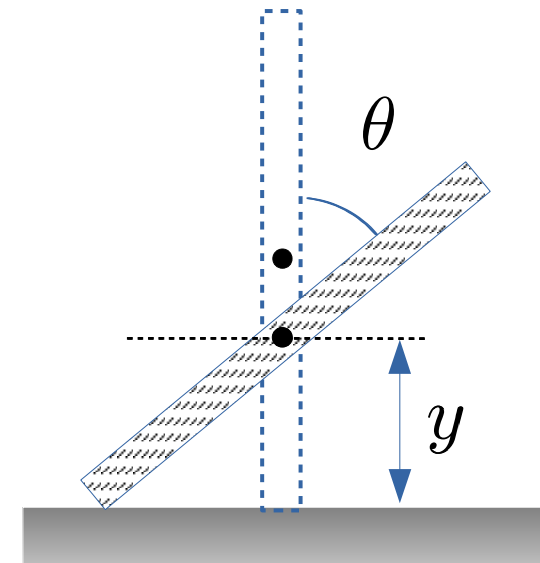
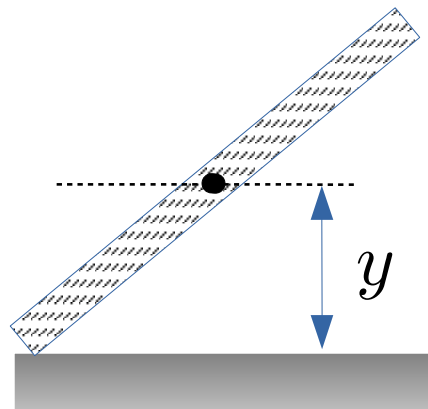
مکانیک تحلیلی

محمدرضا مظفری
گروه فیزیک، دانشکده علوم پایه
دانشگاه قم
اسفند ۹۸

حرکت اجسام صلب در صفحه

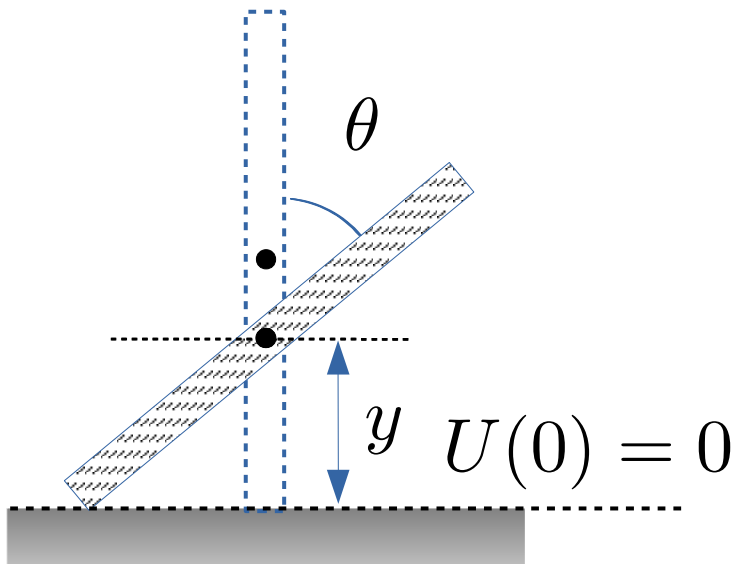


بدون اصطکاک



$$y = \frac{L}{2} \cos \theta$$

حرکت اجسام صلب در صفحه



$$E_1 = Mg \frac{L}{2}$$

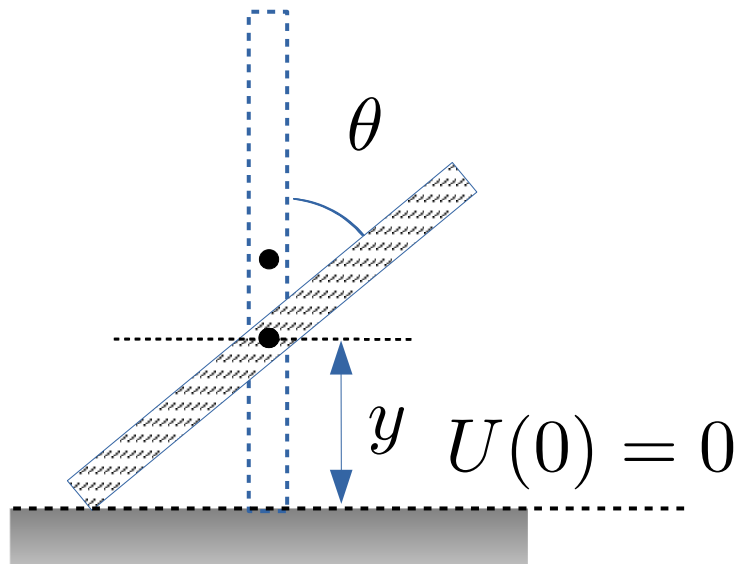
$$E_2 = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} \mathbb{I}_{\text{CM}} \dot{\theta}^2 + Mgy$$

$$\mathbb{I}_{\text{CM}} = \frac{1}{12} ML^2$$

$$\frac{d}{dt} y = \frac{d}{dt} \frac{L}{2} \cos \theta \Rightarrow \dot{y} = -\frac{L}{2} \dot{\theta} \sin \theta$$

$$\dot{\theta} = -\frac{2}{L \sin \theta} \dot{y}$$

حرکت اجسام صلب در صفحه



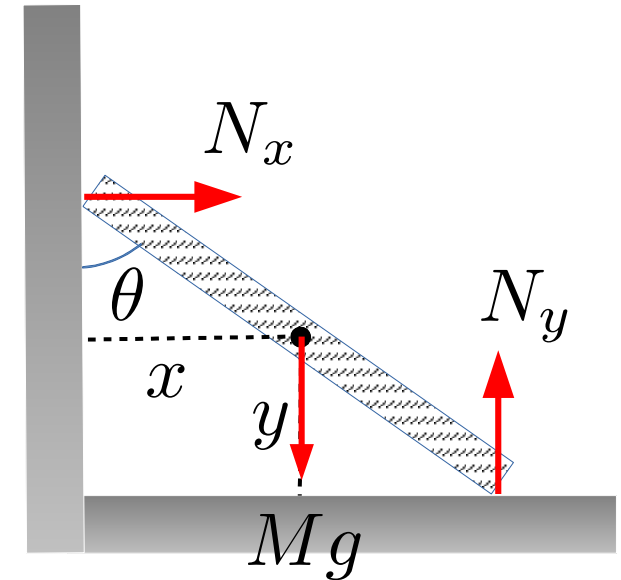
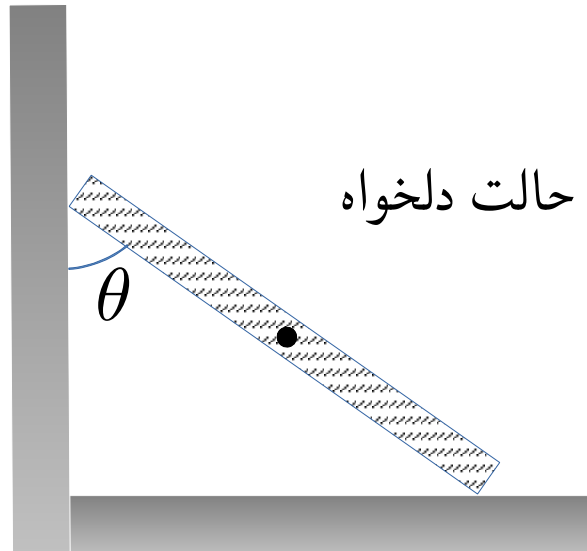
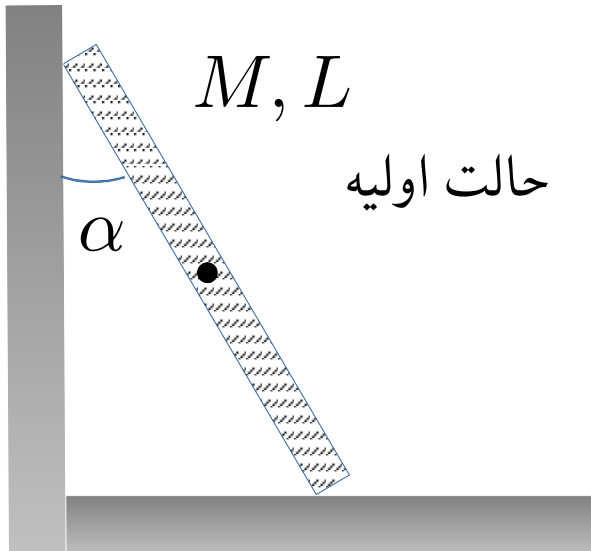
$$\frac{1}{2} M \dot{y}^2 + \frac{1}{2} I_{\text{CM}} \dot{\theta}^2 + Mgy = Mg \frac{L}{2}$$

$$\frac{1}{2} M \left(1 + \frac{1}{3 \sin^2 \theta} \right) \dot{y}^2 = Mg \left(\frac{L}{2} - y \right)$$

$$\frac{1}{2} M \left(1 + \frac{1}{3 \sin^2 \theta} \right) \dot{y}^2 = Mg \left(\frac{L}{2} - y \right)$$

$$\dot{y} = \sqrt{\frac{3g(L - 2y) \sin^2 \theta}{1 + 3 \sin^2 \theta}}$$

حرکت اجسام صلب در صفحه

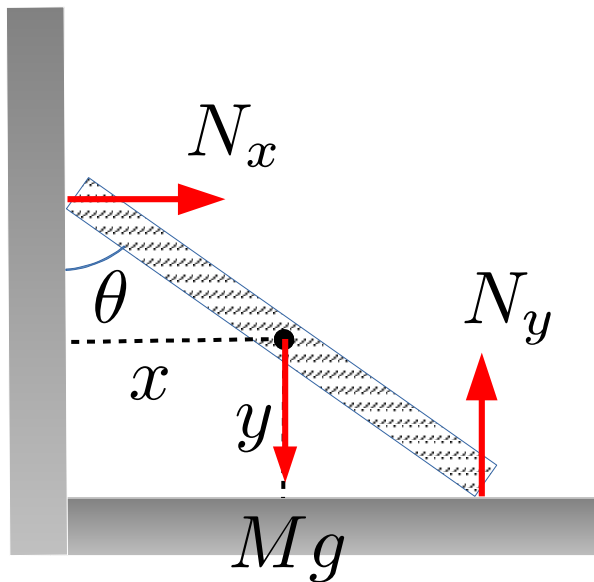


سطوح بدون اصطکاک اند.

$$N_x, N_y = ?$$

میله در چه زاویه‌ای دیوار را ترک می‌کند؟

حرکت اجسام صلب در صفحه



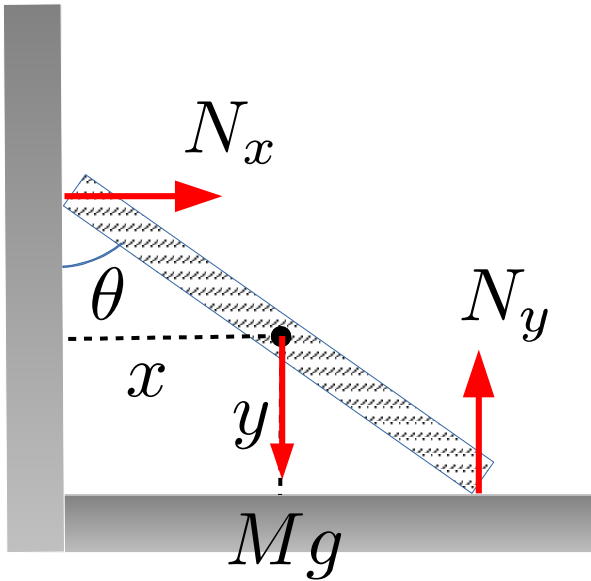
$$\begin{cases} x = \frac{L}{2} \sin \theta \\ y = \frac{L}{2} \cos \theta \end{cases} \quad \begin{cases} \dot{x} = \dot{\theta} \frac{L}{2} \cos \theta \\ \dot{y} = -\dot{\theta} \frac{L}{2} \sin \theta \end{cases}$$

$$\begin{cases} \ddot{x} = \ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta \\ \ddot{y} = -\ddot{\theta} \frac{L}{2} \sin \theta - \dot{\theta}^2 \frac{L}{2} \cos \theta \end{cases}$$

$$\begin{cases} N_x = M\ddot{x} \\ N_y - Mg = M\ddot{y} \end{cases} \Rightarrow \begin{cases} N_x = M(\ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta) \\ N_y - Mg = -M(\ddot{\theta} \frac{L}{2} \sin \theta + \dot{\theta}^2 \frac{L}{2} \cos \theta) \end{cases}$$

$$\dot{\theta}, \ddot{\theta} = ?$$

حرکت اجسام صلب در صفحه



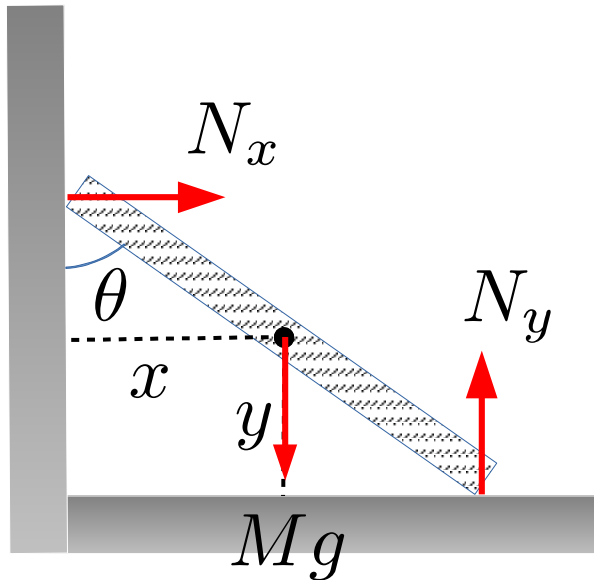
$$\begin{cases} N_x = M(\ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta) \\ N_y - Mg = -M(\ddot{\theta} \frac{L}{2} \sin \theta + \dot{\theta}^2 \frac{L}{2} \cos \theta) \end{cases}$$

$$E_1 = Mg \frac{L}{2} \cos \alpha$$

$$E_2 = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \frac{1}{12} ML^2 \dot{\theta}^2 + Mg \frac{L}{2} \cos \theta$$

$$Mg \frac{L}{2} (\cos \alpha - \cos \theta) = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M \frac{1}{12} ML^2 \dot{\theta}^2$$

حرکت اجسام صلب در صفحه



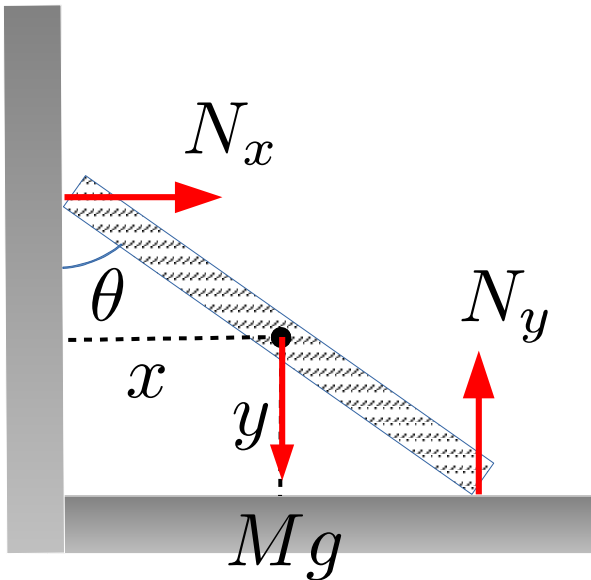
$$\begin{cases} N_x = M(\ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta) \\ N_y - Mg = -M(\ddot{\theta} \frac{L}{2} \sin \theta + \dot{\theta}^2 \frac{L}{2} \cos \theta) \end{cases}$$

$$Mg \frac{L}{2} (\cos \alpha - \cos \theta) = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M \frac{1}{12} M L^2 \dot{\theta}^2$$

$$\begin{cases} \dot{x} = \dot{\theta} \frac{L}{2} \cos \theta \\ \dot{y} = -\dot{\theta} \frac{L}{2} \sin \theta \end{cases} \Rightarrow (\dot{x}^2 + \dot{y}^2) = \frac{L^2}{4} \dot{\theta}^2$$

$$Mg \frac{L}{2} (\cos \alpha - \cos \theta) = \frac{1}{2} M \frac{L^2}{3} \dot{\theta}^2$$

حرکت اجسام صلب در صفحه



$$\begin{cases} N_x = M(\ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta) \\ N_y - Mg = -M(\ddot{\theta} \frac{L}{2} \sin \theta + \dot{\theta}^2 \frac{L}{2} \cos \theta) \end{cases}$$

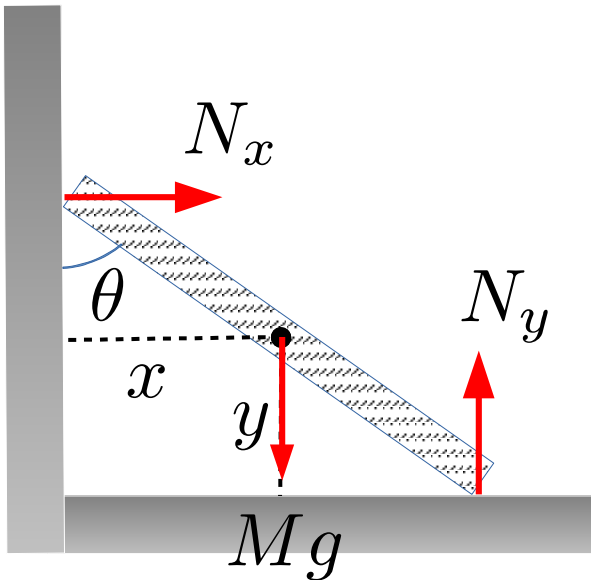
$$Mg \frac{L}{2} (\cos \alpha - \cos \theta) = \frac{1}{2} M \frac{L^2}{3} \dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{3g}{L} (\cos \alpha - \cos \theta)$$

$$\frac{d}{dt} \dot{\theta}^2 = \frac{d}{dt} \frac{3g}{L} (\cos \alpha - \cos \theta) \Rightarrow 2\dot{\theta}\ddot{\theta} = \frac{3g}{L} \dot{\theta} \sin \theta$$

$$\ddot{\theta} = \frac{3g}{2L} \sin \theta$$

حرکت اجسام صلب در صفحه



$$\begin{cases} N_x = M(\ddot{\theta} \frac{L}{2} \cos \theta - \dot{\theta}^2 \frac{L}{2} \sin \theta) \\ N_y - Mg = -M(\ddot{\theta} \frac{L}{2} \sin \theta + \dot{\theta}^2 \frac{L}{2} \cos \theta) \end{cases}$$

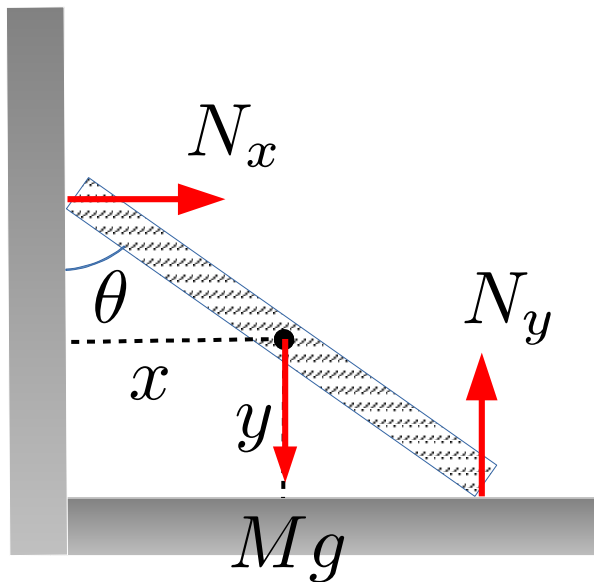
$$\dot{\theta}^2 = \frac{3g}{L} (\cos \alpha - \cos \theta)$$

$$\ddot{\theta} = \frac{3g}{2L} \sin \theta$$

$$N_x(\theta) = \frac{3}{2} Mg \sin \theta \left(\frac{3}{2} \cos \theta - \cos \alpha \right)$$

$$N_y(\theta) = Mg \left(1 - \frac{3}{4} \sin^2 \theta - \frac{3}{2} (\cos \alpha - \cos \theta) \cos \theta \right)$$

حرکت اجسام صلب در صفحه



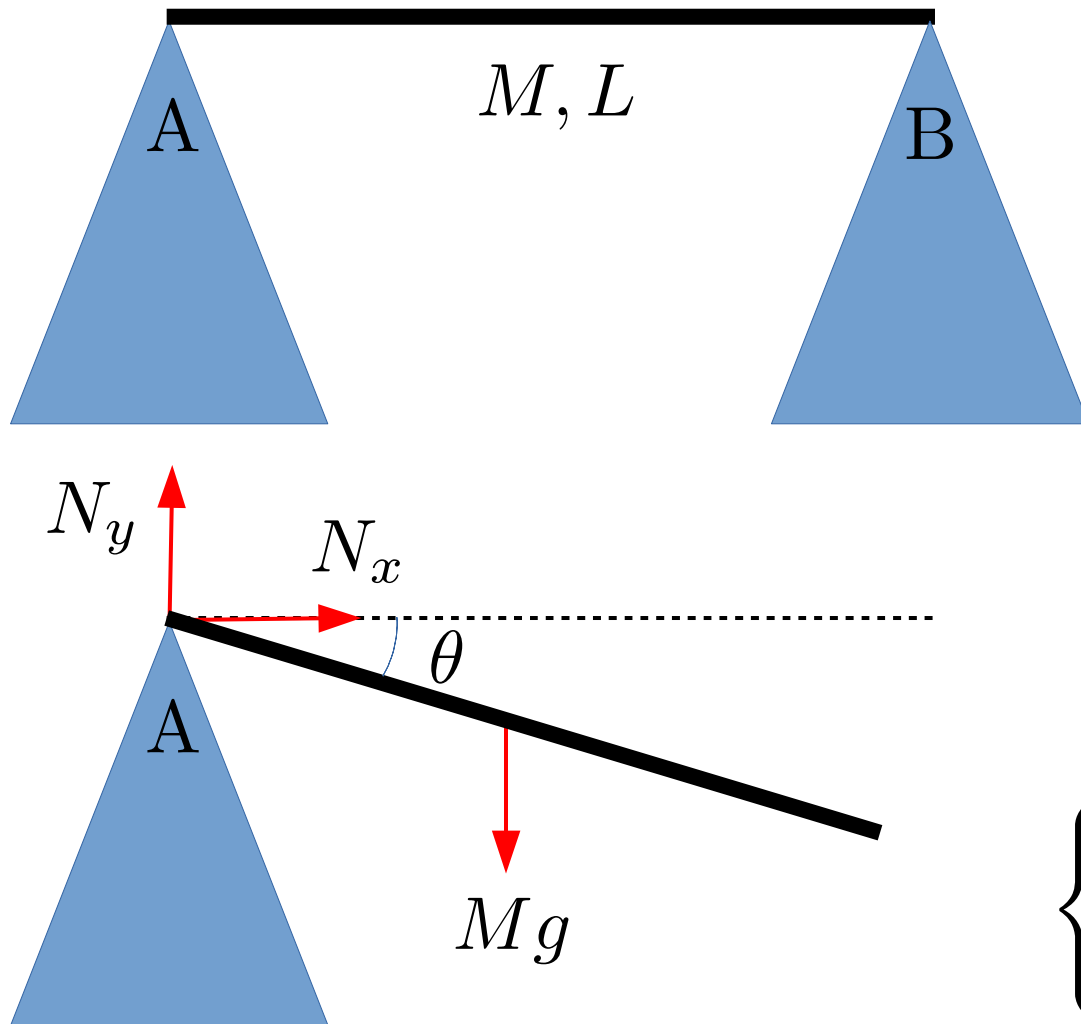
$$N_x(\theta) = \frac{3}{2} M g \sin \theta \left(\frac{3}{2} \cos \theta - \cos \alpha \right)$$

$$N_y(\theta) = M g \left(1 - \frac{3}{4} \sin^2 \theta - \frac{3}{2} (\cos \alpha - \cos \theta) \cos \theta \right)$$

$$N_x(\beta) = 0 \Rightarrow \frac{3}{2} \cos \beta - \cos \alpha = 0 \Rightarrow \cos \beta = \frac{2}{3} \cos \alpha$$

$$\beta = \cos^{-1} \left(\frac{2}{3} \cos \alpha \right)$$

حرکت اجسام صلب در صفحه

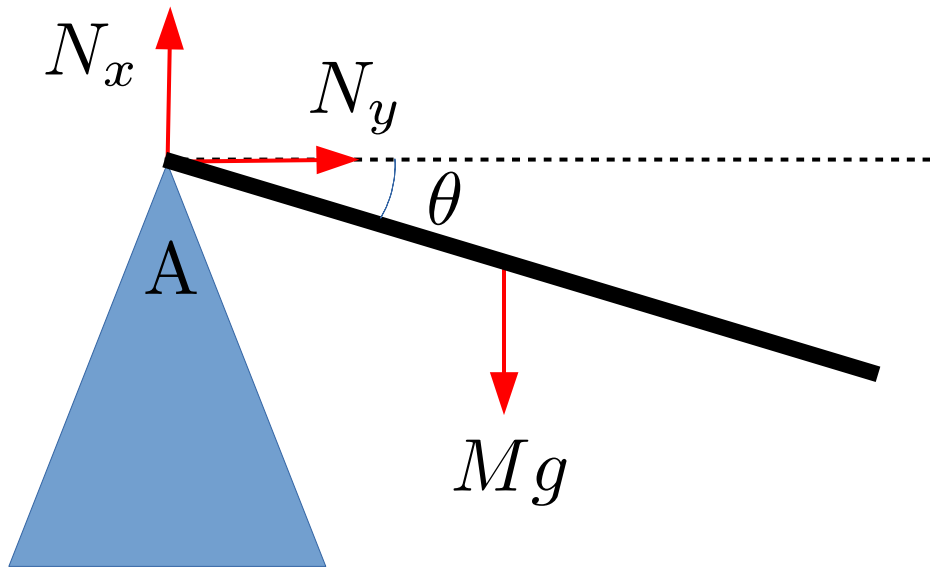


$$\begin{cases} x = \frac{L}{2} \cos \theta \\ y = -\frac{L}{2} \sin \theta \end{cases}$$

$$\begin{cases} \dot{x} = -\dot{\theta} \frac{L}{2} \sin \theta \\ \dot{y} = -\dot{\theta} \frac{L}{2} \cos \theta \end{cases}$$

$$\begin{cases} \ddot{x} = -\ddot{\theta} \frac{L}{2} \sin \theta - \dot{\theta}^2 \frac{L}{2} \cos \theta \\ \ddot{y} = -\ddot{\theta} \frac{L}{2} \cos \theta + \dot{\theta}^2 \frac{L}{2} \sin \theta \end{cases}$$

حرکت اجسام صلب در صفحه

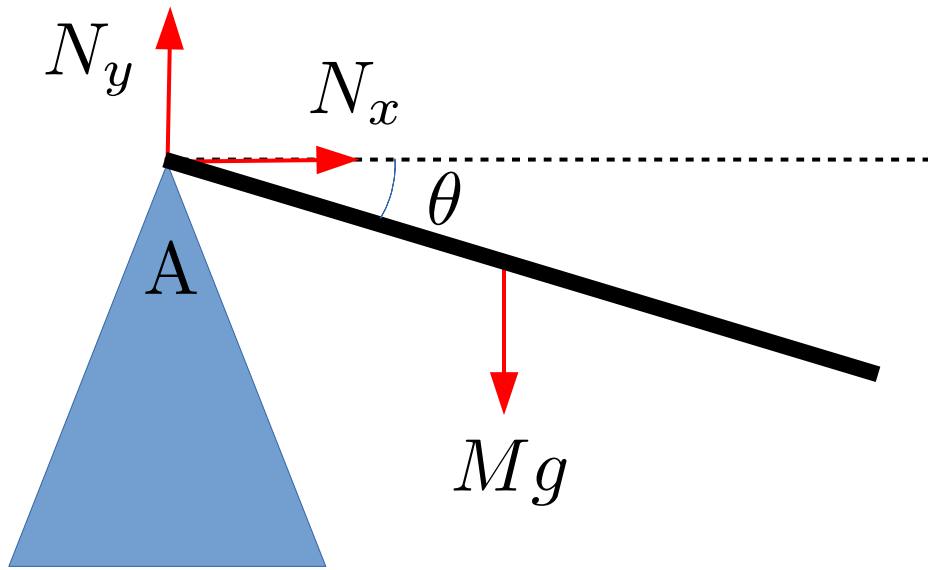


$$\begin{cases} \ddot{x} = -\ddot{\theta} \frac{L}{2} \sin \theta - \dot{\theta}^2 \frac{L}{2} \cos \theta \\ \ddot{y} = -\ddot{\theta} \frac{L}{2} \cos \theta + \dot{\theta}^2 \frac{L}{2} \sin \theta \end{cases}$$

$$\begin{cases} N_x = M\ddot{x} \\ N_y - Mg = M\ddot{y} \end{cases}$$

$$\begin{cases} N_x = -M \frac{L}{2} (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \\ N_y - Mg = M \frac{L}{2} (-\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) \end{cases}$$

حرکت اجسام صلب در صفحه



$$E_1 = 0$$

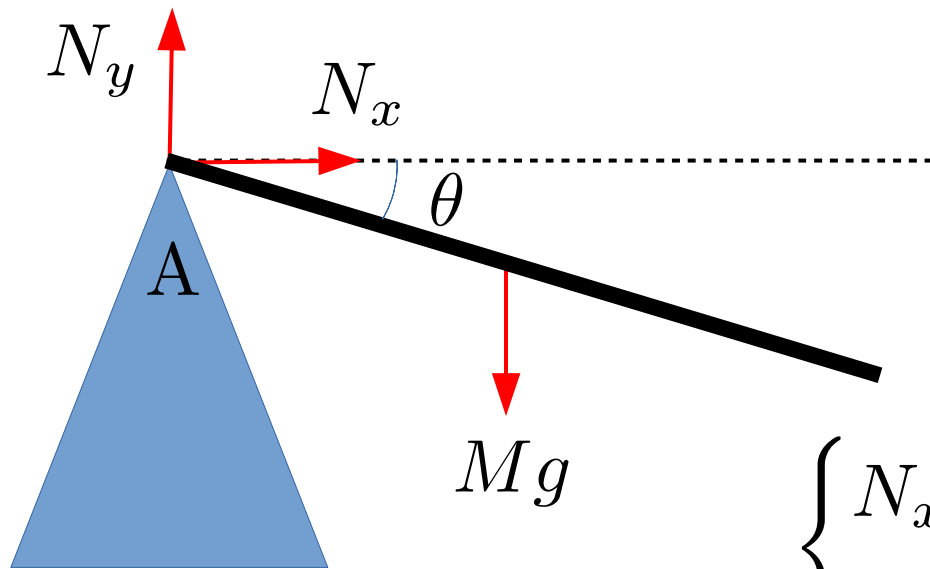
$$E_2 = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2)$$

$$+ \frac{1}{2} \frac{1}{12} M L^2 \dot{\theta}^2 - M g \frac{L}{2} \sin \theta$$

$$0 = \frac{1}{6} M L^2 \dot{\theta}^2 - M g \frac{L}{2} \sin \theta \Rightarrow \dot{\theta}^2 = \frac{3g}{L} \sin \theta$$

$$\frac{d}{dt} \dot{\theta}^2 = \frac{d}{dt} \frac{3g}{L} \sin \theta \Rightarrow \ddot{\theta} = \frac{3g}{2L} \cos \theta$$

حرکت اجسام صلب در صفحه



$$\dot{\theta}^2 = \frac{3g}{L} \sin \theta$$

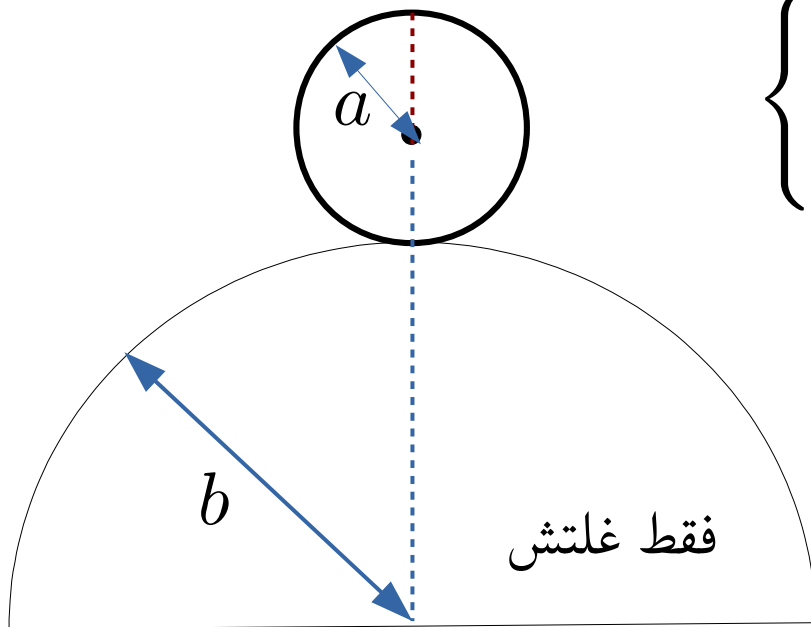
$$\ddot{\theta} = \frac{3g}{2L} \cos \theta$$

$$\begin{cases} N_x = -M \frac{L}{2} (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \\ N_y - Mg = M \frac{L}{2} (-\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta) \end{cases}$$

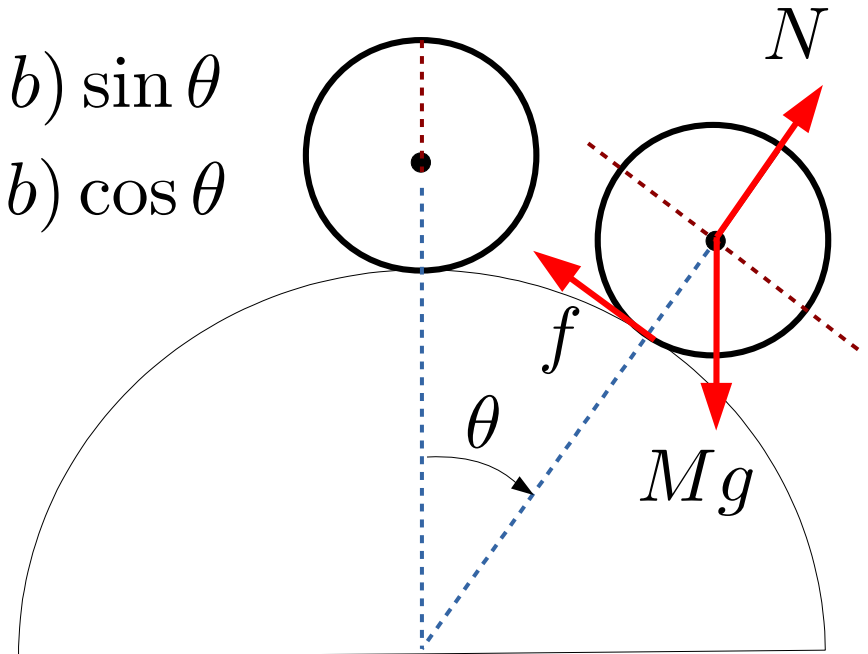
$$N_x = -\frac{9}{4} Mg \sin \theta \cos \theta$$

$$N_y = \frac{1}{4} Mg (1 + 9 \sin^2 \theta)$$

حرکت اجسام صلب در صفحه

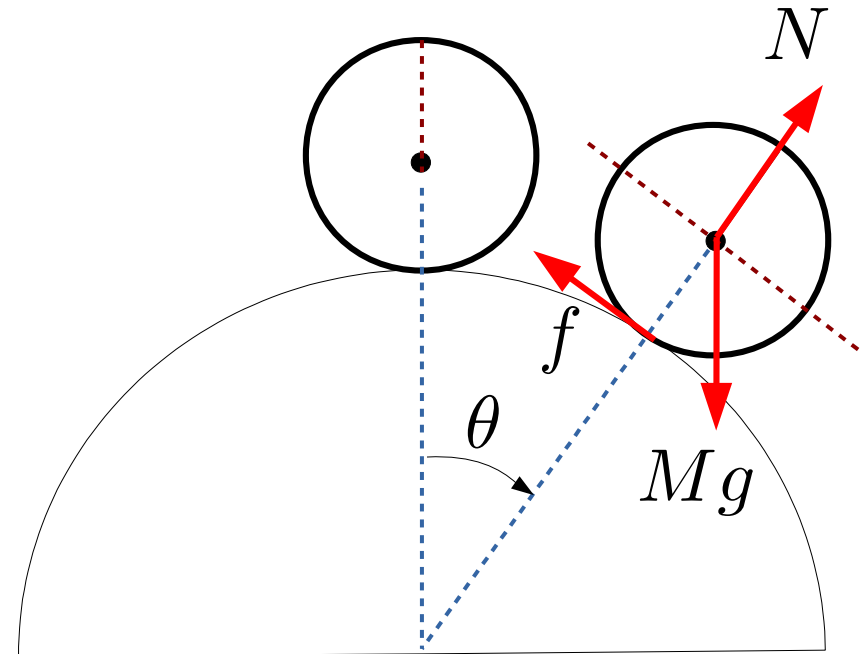
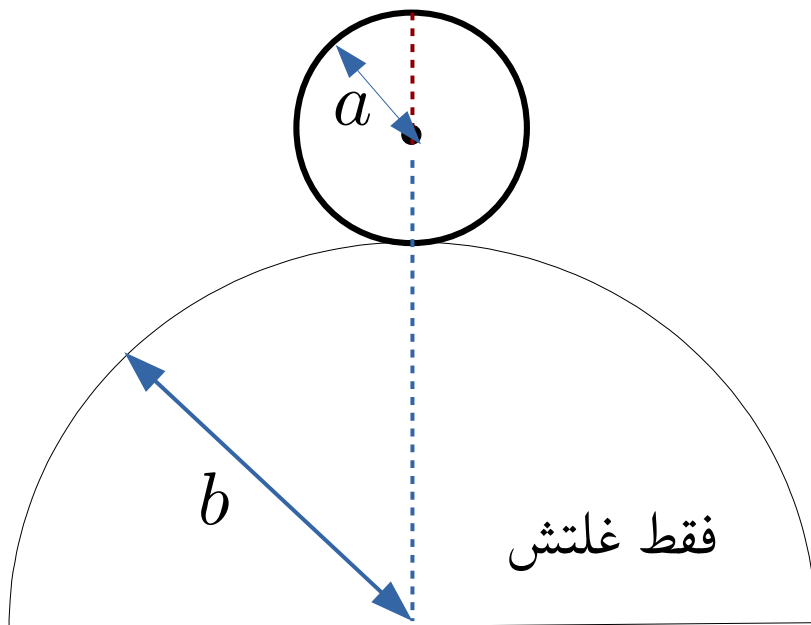


$$\begin{cases} x = (a + b) \sin \theta \\ y = (a + b) \cos \theta \end{cases}$$



$$\begin{cases} \dot{x} = (a + b)\dot{\theta} \cos \theta \\ \dot{y} = -(a + b)\dot{\theta} \sin \theta \end{cases} \Rightarrow \begin{cases} \ddot{x} = (a + b)\ddot{\theta} \cos \theta - (a + b)\dot{\theta}^2 \sin \theta \\ \ddot{y} = -(a + b)\ddot{\theta} \sin \theta - (a + b)\dot{\theta}^2 \cos \theta \end{cases}$$

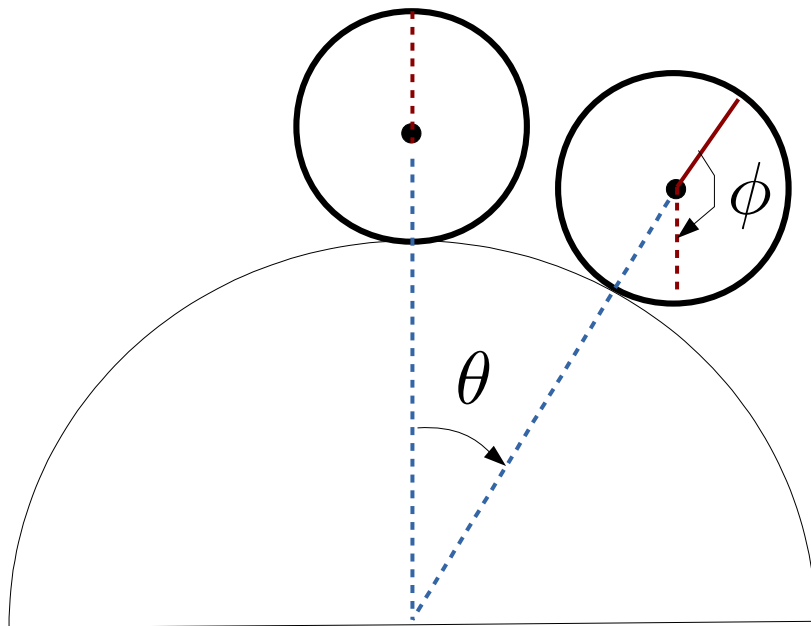
حرکت اجسام صلب در صفحه



$$N - Mg \cos \theta = -M(a + b)\dot{\theta}^2 \Rightarrow N = Mg \cos \theta - M(a + b)\dot{\theta}^2$$

$$Mg \sin \theta - f = M(a + b)\ddot{\theta} \Rightarrow f = Mg \sin \theta - M(a + b)\ddot{\theta}$$

حرکت اجسام صلب در صفحه



$$E_1 = Mg(a + b)$$

$$E_2 = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\dot{\phi}^2 + Mg(a + b)\cos\theta$$

$$\frac{3}{4}M(a + b)^2\dot{\theta}^2 = Mg(a + b)(1 - \cos\theta)$$

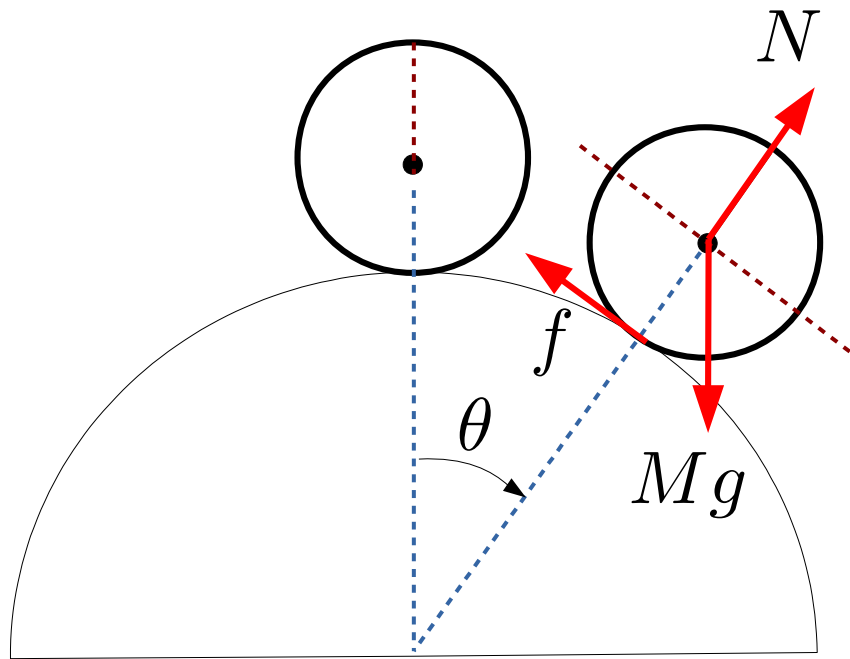
$$(a + b)\theta = a\phi$$

$$(a + b)\dot{\theta} = a\dot{\phi}$$

$$\dot{\theta}^2 = \frac{4g}{3(a + b)}(1 - \cos\theta)$$

$$\ddot{\theta} = \frac{2g}{3(a + b)}\sin\theta$$

حرکت اجسام صلب در صفحه



$$\dot{\theta}^2 = \frac{4g}{3(a+b)} (1 - \cos \theta)$$

$$\ddot{\theta} = \frac{2g}{3(a+b)} \sin \theta$$

$$f = Mg \sin \theta - M(a+b)\ddot{\theta}$$

$$f(\theta) = \frac{1}{3} Mg \sin \theta$$

$$N = Mg \cos \theta - M(a+b)\dot{\theta}^2$$

$$N(\theta = \beta) = 0 \Rightarrow \cos \beta = \frac{4}{7}$$

$$N(\theta) = Mg \left(\frac{7}{3} \cos \theta - \frac{4}{3} \right)$$

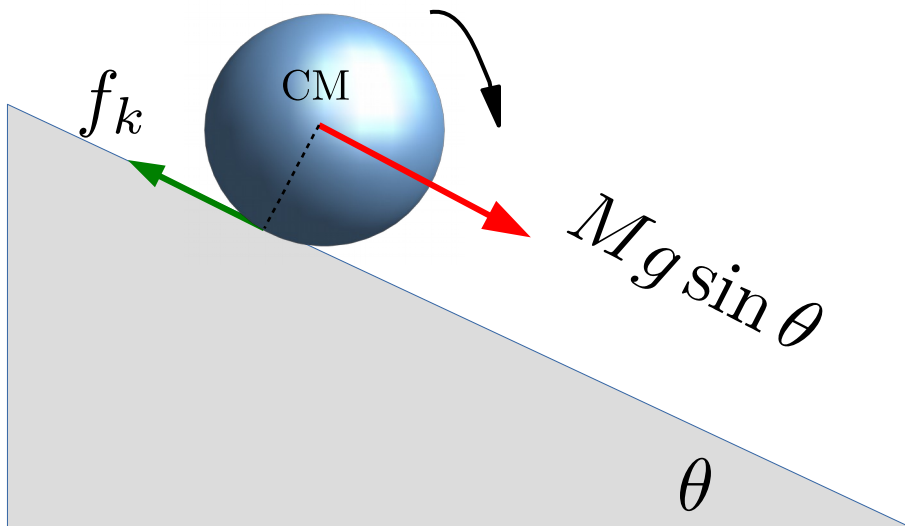
$$\beta = \cos^{-1} \left(\frac{4}{7} \right)$$

حرکت اجسام صلب در صفحه

لغزش و غلتش

$$v \neq R\omega$$

M, R



$$\sum F = Ma$$

$$Mg \sin \theta - \mu_k Mg \cos \theta = Ma$$

$$a = g(\sin \theta - \mu_k \cos \theta)$$

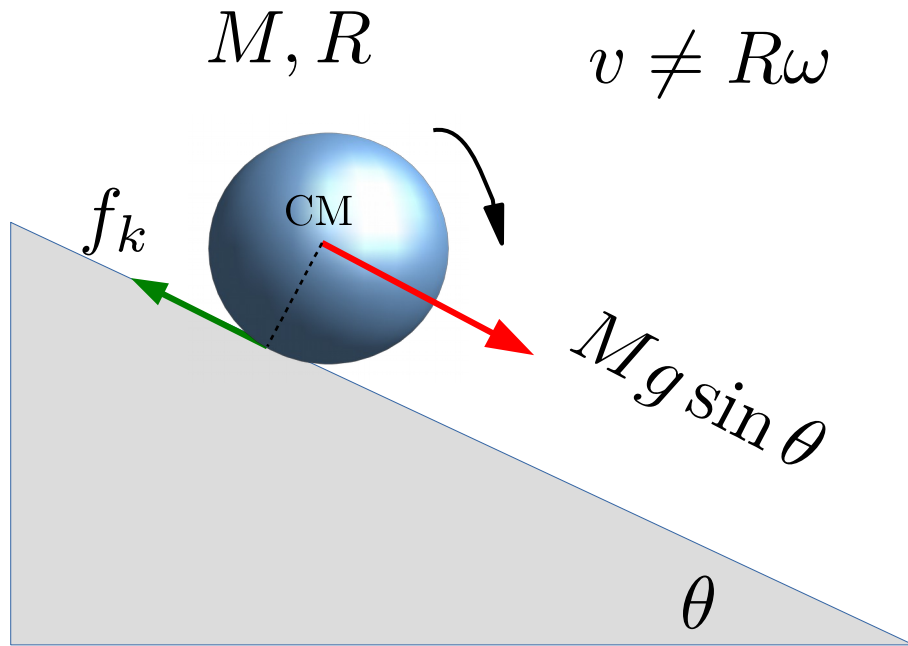
$$v = gt(\sin \theta - \mu_k \cos \theta)$$

$$\omega = \frac{\mu_k MgR \cos \theta}{I_{CM}} t$$

$$\mu_k MgR \cos \theta = I_{CM} \alpha$$

$$\alpha = \frac{\mu_k MgR \cos \theta}{I_{CM}}$$

حرکت اجسام صلب در صفحه



لغزش و غلتش

$$v = gt(\sin \theta - \mu_k \cos \theta)$$

$$\omega = \frac{\mu_k M g R \cos \theta}{\mathbb{I}_{\text{CM}}} t$$

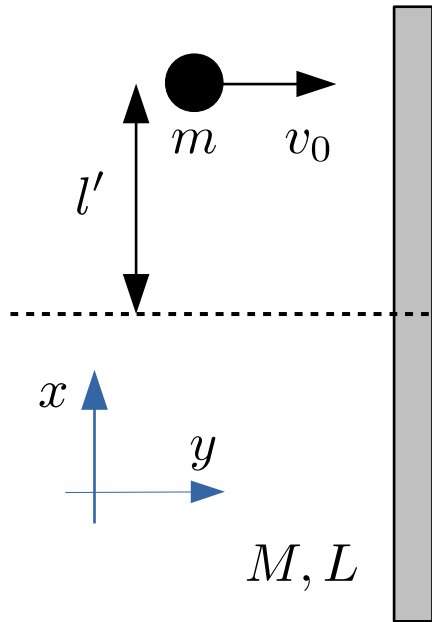
$$v(t) = R\omega(t)$$

$$gt(\sin \theta - \mu_k \cos \theta) = \frac{\mu_k M R^2 \cos \theta}{\mathbb{I}_{\text{CM}}} gt$$

$$\tan \theta = \mu_k \left(1 + \frac{M R^2}{\mathbb{I}_{\text{CM}}} \right) \Rightarrow \mu_k = \frac{\tan \theta}{1 + \frac{M R^2}{\mathbb{I}_{\text{CM}}}}$$

$$\frac{\tan \theta}{1 + \frac{M R^2}{\mathbb{I}_{\text{CM}}}} < \mu_s$$

حرکت اجسام صلب در صفحه



جسم آزاد

پایستگی تکانه خطی:

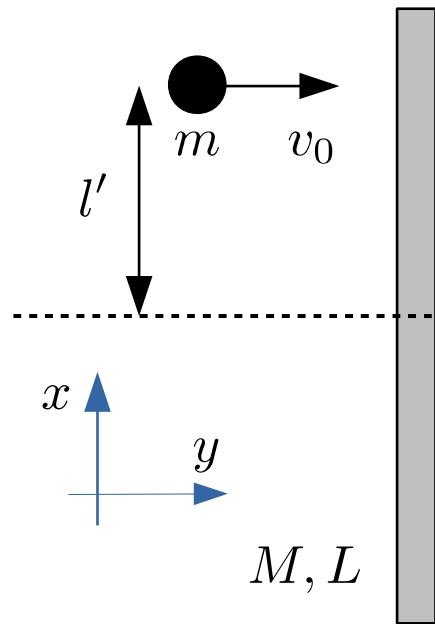
$$mv_0 = mv + MV \Rightarrow mv_0 - mv = MV$$

پایستگی تکانه زاویه‌ای:

$$mv_0 l' = mvl' + \mathbb{I}_{\text{CM}} \omega \Rightarrow mv_0 l' - mvl' = \mathbb{I}_{\text{CM}} \omega$$

$$MV = \frac{\mathbb{I}_{\text{CM}} \omega}{l'} \Rightarrow \int \vec{F} dt = \frac{1}{l'} \int \vec{\tau} dt$$

حرکت اجسام صلب در صفحه



$\otimes g$
ice

جسم آزاد

پایستگی تکانه خطی:

$$mv_0 = mv + MV \Rightarrow mv_0 - mv = MV$$

پایستگی تکانه زاویه‌ای:

$$mv_0 l' = mv l' + \mathbb{I}_{\text{CM}} \omega \Rightarrow mv_0 l' - mv l' = \mathbb{I}_{\text{CM}} \omega$$

$$\int \vec{F} dt = \Delta \vec{p} = MV \hat{y}$$

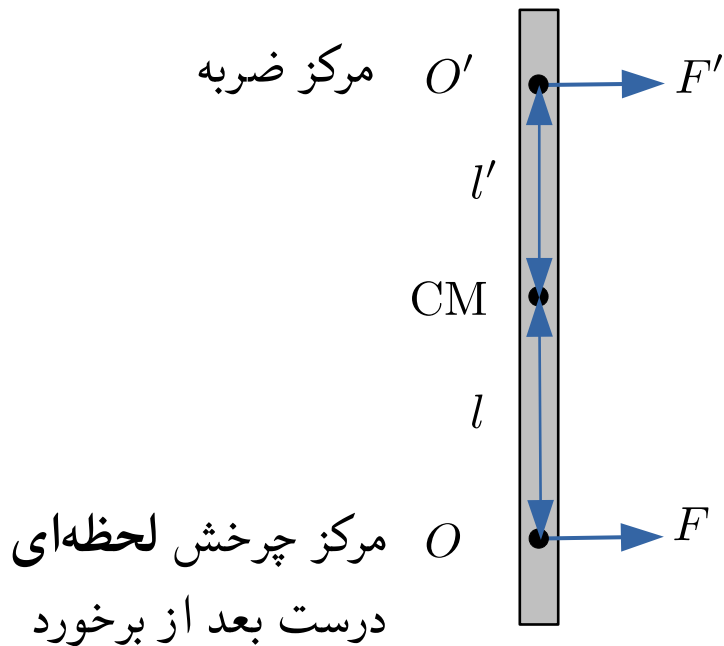
$$\int \vec{\tau} dt = \Delta \vec{L} = \mathbb{I}_{\text{CM}} \omega \hat{z}$$

$$\text{ضربه} = \int \vec{F} dt$$

$$\text{ضربه چرخشی} = \int \vec{\tau} dt$$

حرکت اجسام صلب در صفحه

جسم آزاد



$$MV = \int \vec{F} dt = \int (F + F') dt$$

$$MV = \int F dt + \int F' dt$$

$$\mathbb{I}_O \omega = \int \vec{\tau}_O dt = (l + l') \int F' dt$$

$$\int F' dt = \mathbb{I}_O \omega / (l + l')$$

حرکت اجسام صلب در صفحه

جسم آزاد

مرکز ضربه O'

l'

CM

l

مرکز چرخش لحظه‌ای O

درست بعد از برخورد

F'

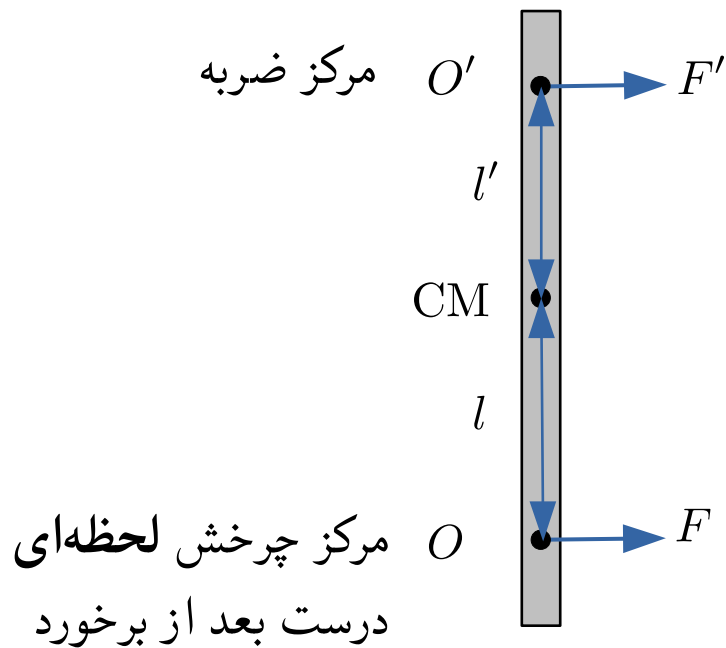
F

$$MV = \int F dt + \int F' dt$$

$$MV = \frac{\mathbb{I}_{CM}\omega}{l'}, \quad \int F' dt = \mathbb{I}_O\omega / (l + l')$$

$$\frac{\mathbb{I}_{CM}\omega}{l'} = \int F dt + \frac{\mathbb{I}_O\omega}{l + l'} \Rightarrow \int F dt = \frac{\mathbb{I}_{CM}\omega}{l'} - \frac{\mathbb{I}_O\omega}{l + l'}$$

حرکت اجسام صلب در صفحه



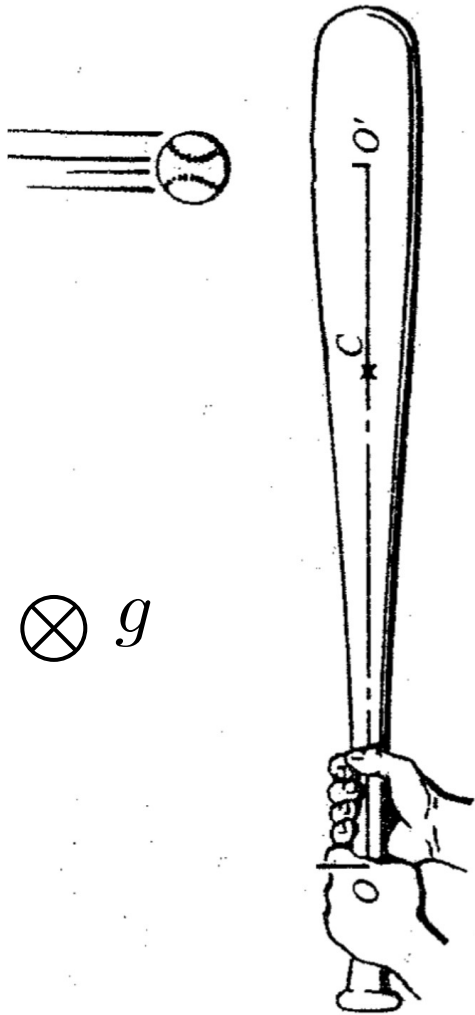
$$\int F dt = \left(\frac{\mathbb{I}_{\text{CM}}}{l'} - \frac{\mathbb{I}_O}{l + l'} \right) \omega$$

$$\mathbb{I}_O = \mathbb{I}_{\text{CM}} + Ml^2$$

$$\int F dt = \left(\frac{\mathbb{I}_{\text{CM}}l - Ml^2l'}{l'(l + l')} \right) \omega$$

حرکت اجسام صلب در صفحه

جسم آزاد



$$\int F dt = \left(\frac{\mathbb{I}_{\text{CM}} l - M l^2 l'}{l'(l + l')} \right) \omega$$

$$\int F dt = 0$$

$$M l l' = \mathbb{I}_{\text{CM}}, \quad \mathbb{I}_{\text{CM}} = M k_{\text{CM}}^2$$

$$l l' = k_{\text{CM}}^2$$