

جلسه دوازدهم

مکانیک تحلیلی

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دانشگاه قم
اسفند ۹۸

انرژی جنبش کل :
$$T = \sum_i^N \left(\frac{1}{2} m_i \dot{x}_i^2 + \frac{1}{2} m_i \dot{y}_i^2 + \frac{1}{2} m_i \dot{z}_i^2 \right)$$

نیروی تعمیم یافته
مختصه k ام :
$$Q_k = \sum_i^N \left(F_i^x \frac{\partial x_i}{\partial q_k} + F_i^y \frac{\partial y_i}{\partial q_k} + F_i^z \frac{\partial z_i}{\partial q_k} \right)$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}, \quad 1 \leq k \leq n$$

معادلات لاگرانژ (در فرم انرژی جنبشی)

مکانیک لاگرانژی

یادآوری

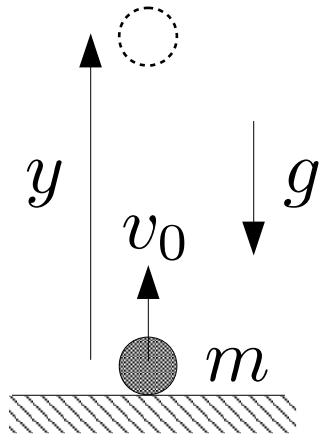
انرژی جنبش کل : $T = \sum_i^N \left(\frac{1}{2} m_i \dot{x}_i^2 + \frac{1}{2} m_i \dot{y}_i^2 + \frac{1}{2} m_i \dot{z}_i^2 \right)$

انرژی پتانسیل کل : $V = V(q_1, q_2, \dots, q_k, \dots, q_n)$

$$\mathcal{L} = T - V$$

معادلات لاگرانژ : $\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_k} \mathcal{L} \right] = \frac{\partial}{\partial q_k} \mathcal{L}, \quad 1 \leq k \leq n$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

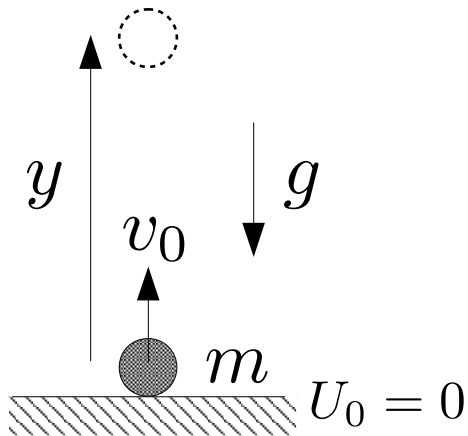
$$q_1 = y, \quad \dot{q}_1 = \dot{y}, \quad F_y = -mg, \quad T = \frac{1}{2} m \dot{y}^2$$

$$Q_y = F_y \frac{dy}{dy} = -mg, \quad \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{y}} \right] = Q_y + \frac{\partial T}{\partial y}$$

$$\frac{d}{dt} [m\dot{y}] = -mg + 0 \Rightarrow \boxed{\ddot{y} = -g}$$

$$\dot{y} = -gt + v_0, \quad y = -\frac{1}{2}gt^2 + v_0t$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right] = \frac{\partial \mathcal{L}}{\partial q_k}$$

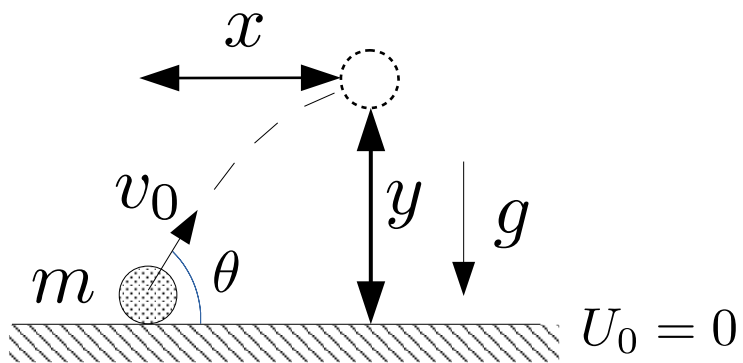
$$q_1 = y, \quad \dot{q}_1 = \dot{y}, \quad T = \frac{1}{2} m \dot{y}^2 \quad V = mgy$$

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{y}^2 - mgy, \quad \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{y}} \right] = \frac{\partial \mathcal{L}}{\partial y}$$

$$\frac{d}{dt} [m\dot{y}] = -mg \Rightarrow \boxed{\ddot{y} = -g}$$

$$\dot{y} = -gt + v_0, \quad y = -\frac{1}{2}gt^2 + v_0t$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

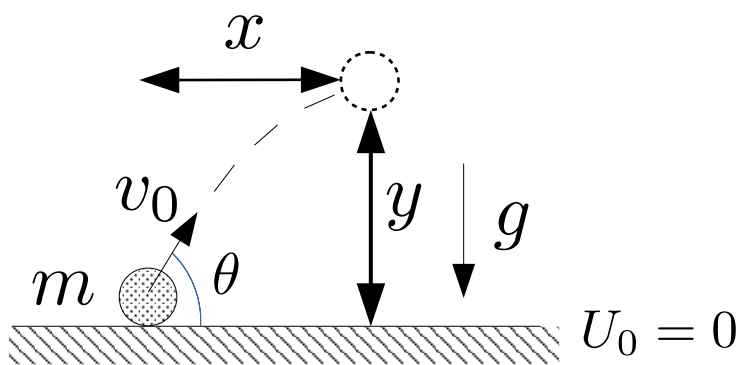
$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad F_x = 0$$

$$q_2 = y, \quad \dot{q}_2 = \dot{y}, \quad F_y = -mg$$

$$Q_y = -mg, \quad Q_x = 0, \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x} \Rightarrow \frac{d}{dt} [m\dot{x}] = 0 \Rightarrow \boxed{\ddot{x} = 0}$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

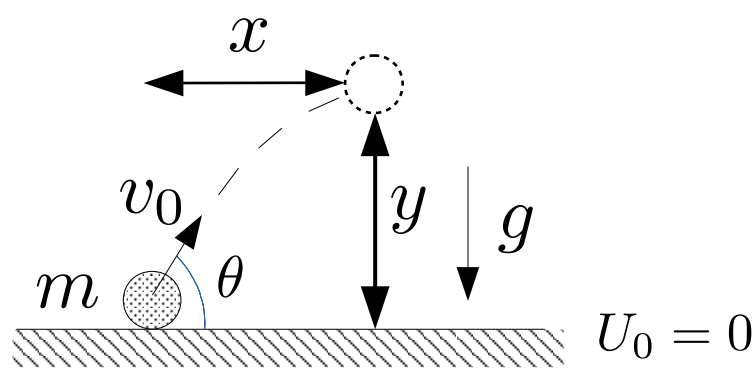
$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad F_x = 0$$

$$q_2 = y, \quad \dot{q}_2 = \dot{y}, \quad F_y = -mg$$

$$Q_y = -mg, \quad Q_x = 0, \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{y}} \right] = Q_y + \frac{\partial T}{\partial y} \Rightarrow \frac{d}{dt} [m\dot{y}] = -mg \Rightarrow \boxed{\ddot{y} = -g}$$

مکانیک لاگرانژی



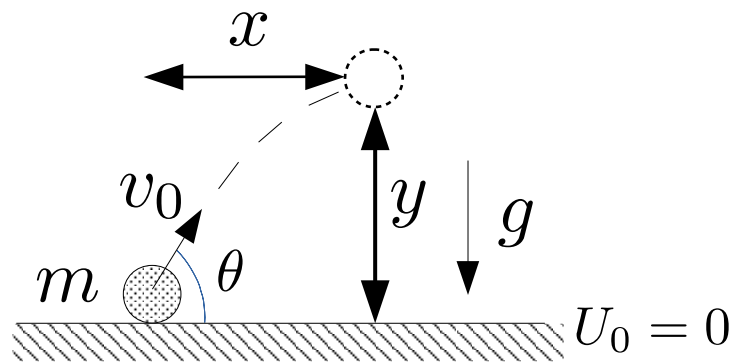
$$\ddot{x} = 0, \quad \ddot{y} = -g$$

$$\vec{v}_0 = v_0 (\cos \theta, \sin \theta)$$

$$x = v_0 t \cos \theta$$

$$y = -\frac{1}{2}gt^2 + v_0 t \sin \theta$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right] = \frac{\partial \mathcal{L}}{\partial q_k}$$

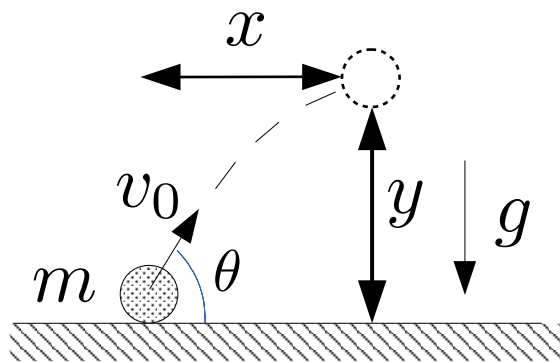
$$q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$q_2 = y, \quad \dot{q}_2 = \dot{y}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2), \quad V = mgy$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

مکانیک لاگرانژی



$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

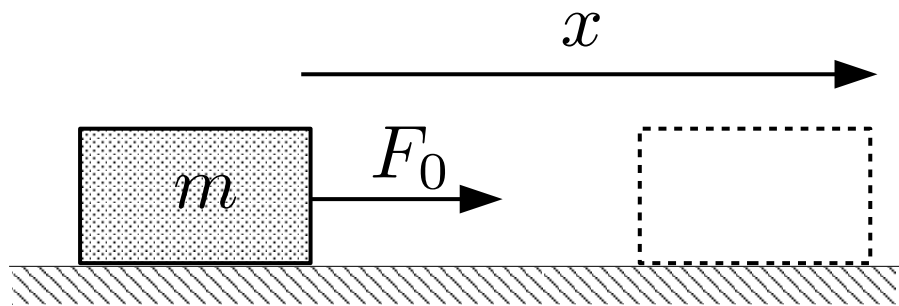
$$U_0 = 0 \quad \frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}} \right] = \frac{\partial \mathcal{L}}{\partial x} \Rightarrow \frac{d}{dt} [m\dot{x}] = 0$$

$$\ddot{x} = 0$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{y}} \right] = \frac{\partial \mathcal{L}}{\partial y} \Rightarrow \frac{d}{dt} [m\dot{y}] = -mg$$

$$\ddot{y} = -g$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

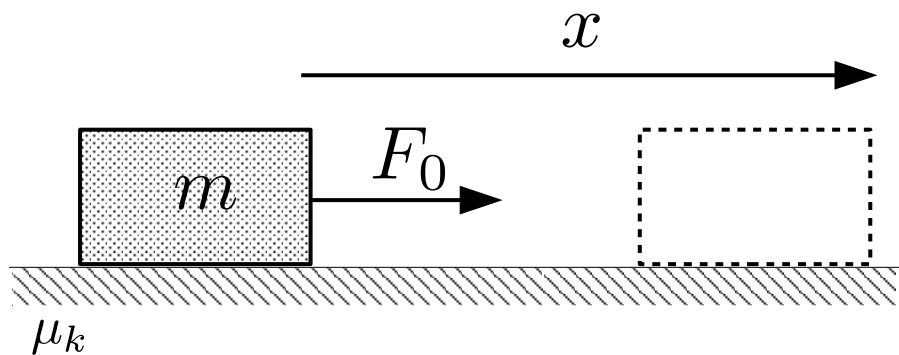
$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad F_x = F_0, \quad T = \frac{1}{2} m \dot{x}^2$$

$$Q_x = F_0, \quad \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m\dot{x}] = F_0 + 0 \Rightarrow m\ddot{x} = F_0$$

$$\ddot{x} = \frac{F_0}{m}$$

مکانیک لاگرانژی



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad F_x = F_0 - \mu_k m g, \quad T = \frac{1}{2} m \dot{x}^2$$

$$Q_x = F_x \frac{dx}{dx} = F_0 - \mu_k m g, \quad \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m \dot{x}] = F_0 - \mu_k m g + 0 \Rightarrow m \ddot{x} = F_0 - \mu_k m g$$

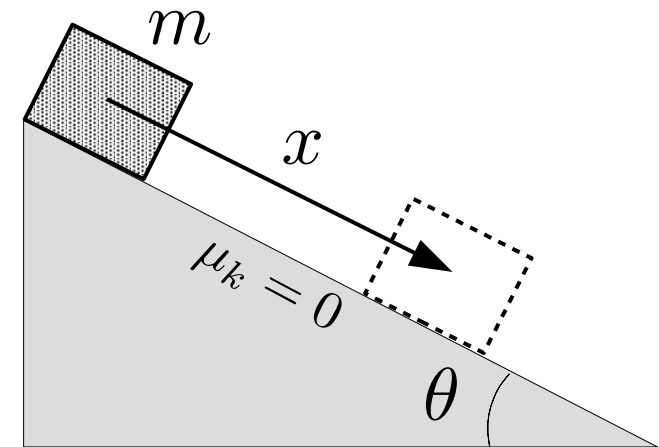
$$\ddot{x} = \frac{F_0}{m} - \mu_k g$$

مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad T = \frac{1}{2} m \dot{x}^2$$

$$Q_x = mg \sin \theta$$



$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m\dot{x}] = mg \sin \theta + 0 \Rightarrow \boxed{\ddot{x} = g \sin \theta}$$

مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_k} \mathcal{L} \right] = \frac{\partial}{\partial q_k} \mathcal{L}$$

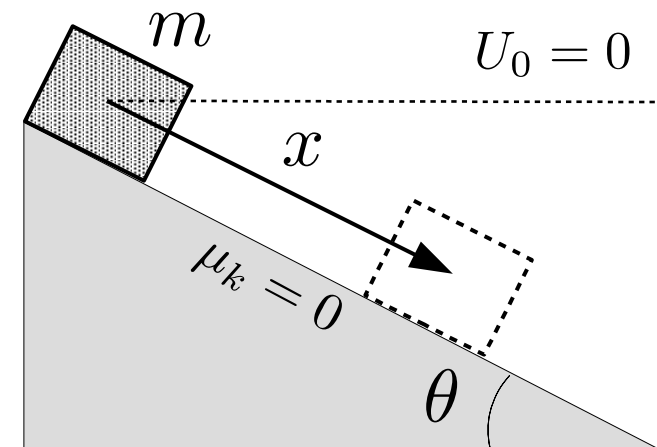
$$q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$T = \frac{1}{2} m \dot{x}^2, \quad V(x) = -mgx \sin \theta$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - (-mgx \sin \theta)$$

$$\frac{d}{dt} [m\dot{x}] = mg \sin \theta \Rightarrow \boxed{\ddot{x} = g \sin \theta}$$



$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L}$$

مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

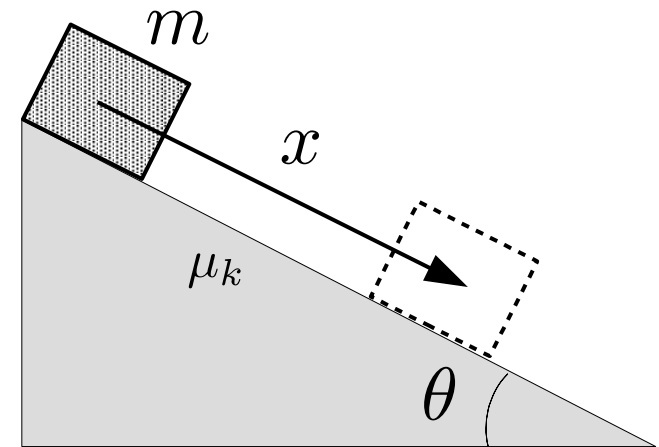
$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad T = \frac{1}{2} m \dot{x}^2$$

$$Q_x = mg \sin \theta - \mu_k mg \cos \theta$$

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m\dot{x}] = mg \sin \theta - \mu_k mg \cos \theta + 0$$

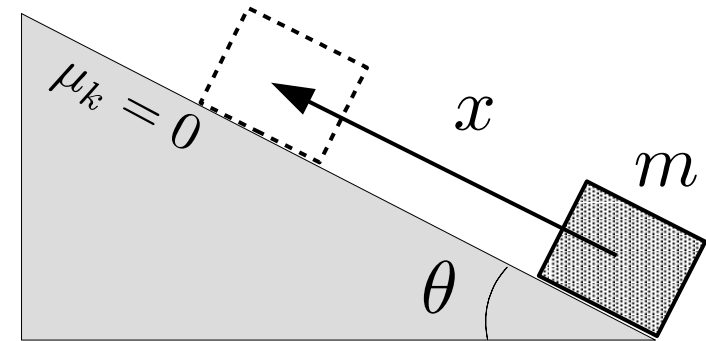
$$\ddot{x} = g(\sin \theta - \mu_k \cos \theta)$$



مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad T = \frac{1}{2} m \dot{x}^2$$



$$Q_x = -mg \sin \theta, \quad \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m\dot{x}] = -mg \sin \theta$$

$$\ddot{x} = -g \sin \theta$$

مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

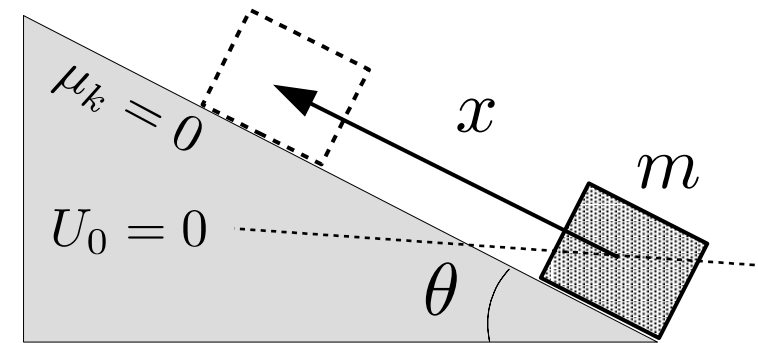
$$q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$T = \frac{1}{2} m \dot{x}^2, \quad V(x) = mgx \sin \theta$$

$$\mathcal{L} = T - V \quad \mathcal{L} = \frac{1}{2} m \dot{x}^2 - mgx \sin \theta$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L}$$

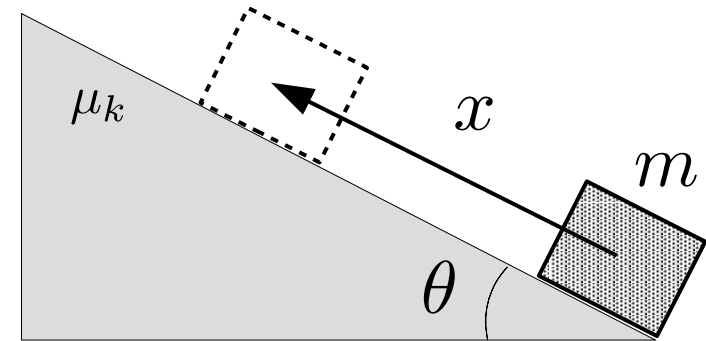
$$\frac{d}{dt} [m\dot{x}] = -mg \sin \theta \Rightarrow \boxed{\ddot{x} = -g \sin \theta}$$



مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}_k} \right] = Q_k + \frac{\partial T}{\partial q_k}$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad T = \frac{1}{2} m \dot{x}^2$$



$$Q_x = -mg \sin \theta - \mu_k mg \cos \theta, \quad \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{x}} \right] = Q_x + \frac{\partial T}{\partial x}$$

$$\frac{d}{dt} [m\dot{x}] = -mg \sin \theta - \mu_k mg \cos \theta$$

$$\ddot{x} = -g(\sin \theta + \mu_k \cos \theta)$$

مکانیک لاگرانژی

$$M : y, \quad \dot{y}$$

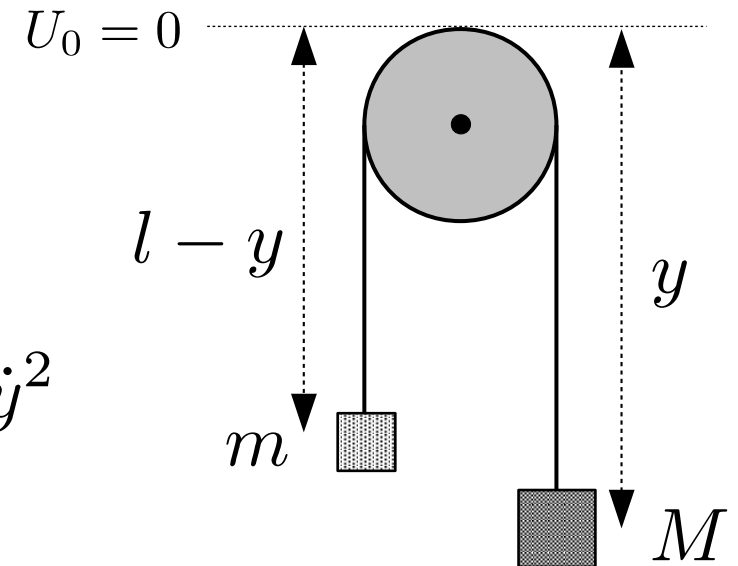
$$m : l - y, \quad -\dot{y}$$

$$q_1 = y, \quad \dot{q}_1 = \dot{y}, \quad T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2$$

$$V = -Mgy - mg(l - y)$$

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 - (-Mgy - mg(l - y))$$

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + Mgy - mgy + c$$



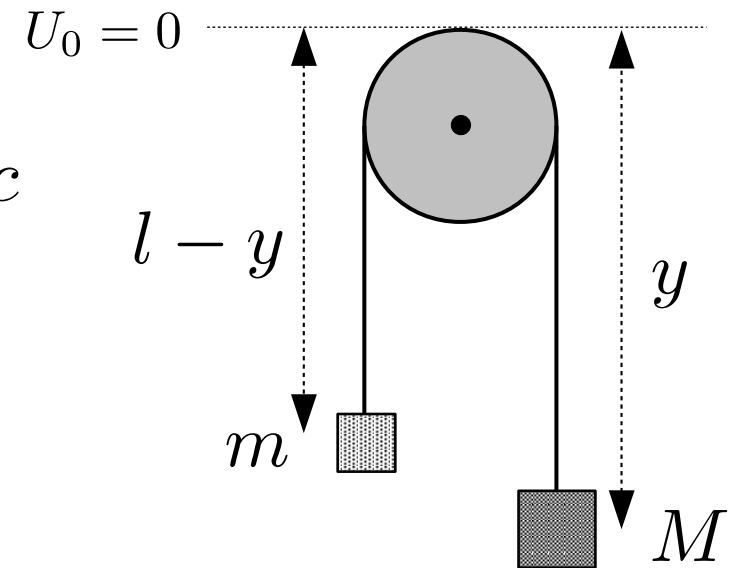
مکانیک لاگرانژی

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + Mgy - mgy + c$$

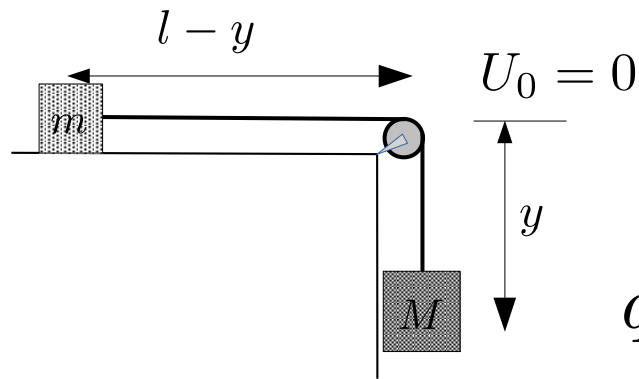
$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{y}} \mathcal{L} \right] = \frac{\partial}{\partial y} \mathcal{L}$$

$$\frac{d}{dt} [(M + m)\dot{y}] = Mg - mg$$

$$\dot{y} = \left(\frac{M - m}{M + m} \right) g$$



مکانیک لاگرانژی



$$M : y, \quad \dot{y}$$

$$m : l - y, \quad -\dot{y}$$

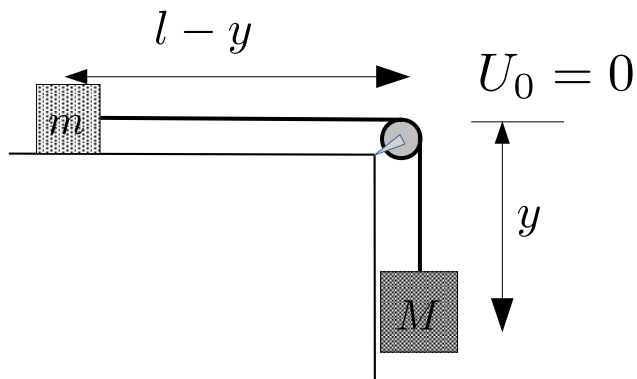
$$q_1 = y, \quad \dot{q}_1 = \dot{y}, \quad T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2$$

$$V = -Mgy$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\dot{y}^2 + Mgy$$

مکانیک لاگرانژی



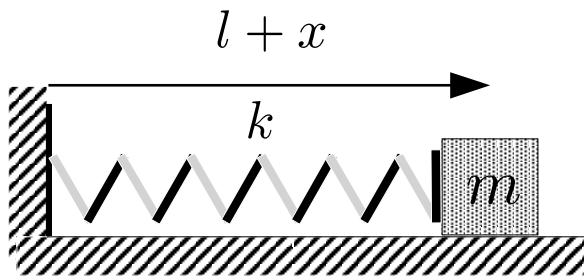
$$\mathcal{L} = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} m \dot{y}^2 + Mgy$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{y}} \mathcal{L} \right] = \frac{\partial}{\partial y} \mathcal{L}$$

$$\frac{d}{dt} [(M + m)\dot{y}] = Mg$$

$$\ddot{y} = \left(\frac{M}{M + m} \right) g$$

مکانیک لاگرانژی



$$m : l + x, \quad \dot{x}$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

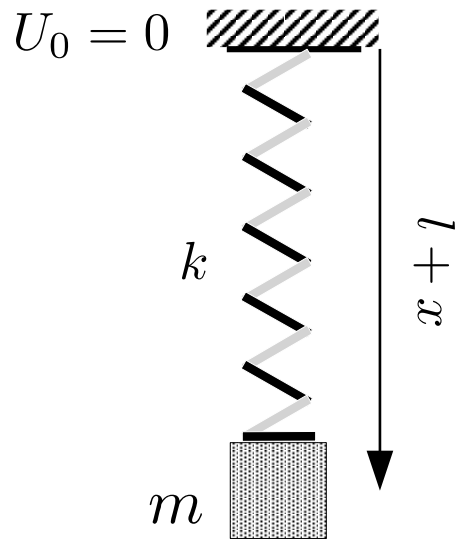
$$q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L} \Rightarrow m\ddot{x} = -kx \Rightarrow \boxed{m\ddot{x} + kx = 0}$$

$$\ddot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$x(t) = A \cos(\omega_0 t + \phi)$$

مکانیک لاگرانژی



$$m : l + x, \quad \dot{x}$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mg(l + x)$$

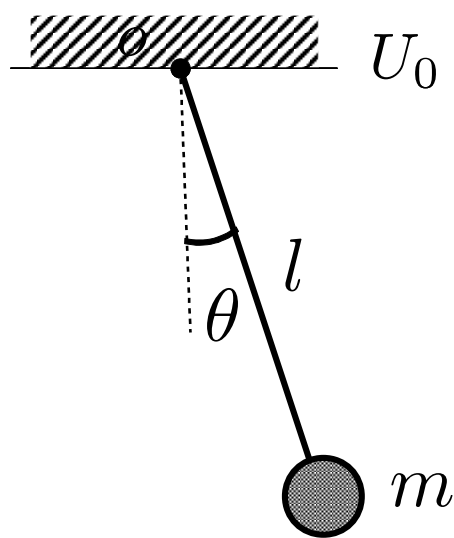
$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mgx + c$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L} \Rightarrow m\ddot{x} = -kx + mg$$

$$m\ddot{x} + kx = mg$$

مکانیک لاگرانژی



$$U_0 = 0$$

$$q_1 = \theta, \quad \dot{q}_1 = \dot{\theta}$$

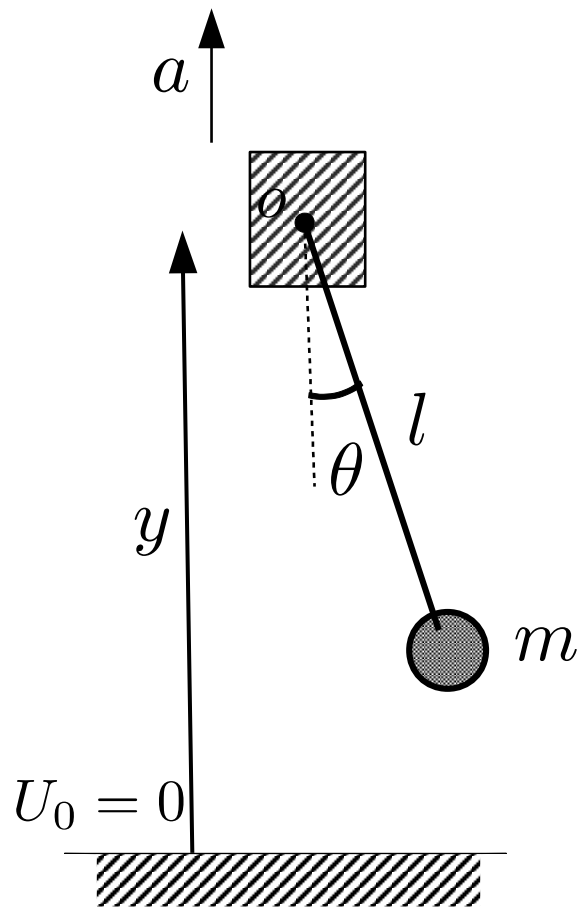
$$T = \frac{1}{2} \mathbb{I}_o \dot{\theta}^2 = \frac{1}{2} m l^2 \dot{\theta}^2, \quad V(\theta) = -mgl \cos \theta$$

$$\mathcal{L} = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right] = \frac{\partial \mathcal{L}}{\partial q_k}$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] = \frac{\partial \mathcal{L}}{\partial \theta} \Rightarrow m l^2 \ddot{\theta} = -mgl \sin \theta \Rightarrow \ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

مکانیک لاگرانژی



$$m : (l \sin \theta, y - l \cos \theta) \quad y = \frac{1}{2}at^2$$

$$m : (l\dot{\theta} \cos \theta, \dot{y} + l\dot{\theta} \sin \theta) \quad \dot{y} = at$$

$$q_1 = \theta, \quad \dot{q}_1 = \dot{\theta}$$

$$T = \frac{1}{2}m(a^2t^2 + l^2\dot{\theta}^2 + 2lat\dot{\theta} \sin \theta)$$

$$V = mg(y - l \cos \theta) = mg \left(\frac{1}{2}at^2 - l \cos \theta \right)$$

$$\mathcal{L} = \frac{1}{2}m(a^2t^2 + l^2\dot{\theta}^2 + 2lat\dot{\theta} \sin \theta)$$

$$-mg \left(\frac{1}{2}at^2 - l \cos \theta \right)$$

مکانیک لاگرانژی

$$\mathcal{L} = \frac{1}{2}m(a^2t^2 + l^2\dot{\theta}^2 + 2lat\dot{\theta} \sin \theta) - mg \left(\frac{1}{2}at^2 - l \cos \theta \right)$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right] = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\frac{d}{dt} \left[ml^2\dot{\theta} + ml at \sin \theta \right] = ml at \dot{\theta} \cos \theta - mgl \sin \theta$$

$$ml^2\ddot{\theta} + mla \sin \theta + \cancel{ml at \dot{\theta} \cos \theta} = \cancel{ml at \dot{\theta} \cos \theta} - mgl \sin \theta$$

$$ml^2\ddot{\theta} + mla \sin \theta = -mgl \sin \theta$$

مکانیک لاگرانژی

$$ml^2\ddot{\theta} + mla \sin \theta = -mgl \sin \theta$$

$$ml^2\ddot{\theta} + ml(a + g) \sin \theta = 0$$

$$\xrightarrow{\div ml^2} \ddot{\theta} + \frac{a + g}{l} \sin \theta = 0$$

برای θ کوچک:

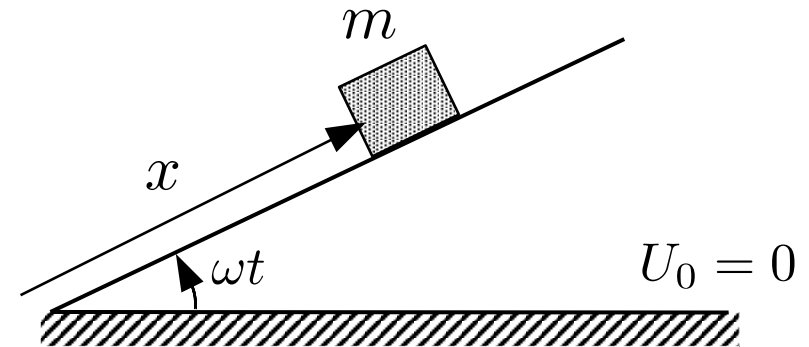
$$\ddot{\theta} + \frac{a + g}{l} \theta = 0$$

$$\omega_0 = \sqrt{\frac{a + g}{l}}$$

مکانیک لاگرانژی

$$m : (x \cos \omega t, x \sin \omega t)$$

$$m : (\dot{x} \cos \omega t - \omega x \sin \omega t, \\ \dot{x} \sin \omega t + x \omega \cos \omega t)$$



$$T = \frac{1}{2} m (\dot{x}^2 + \omega^2 x^2), \quad V = mgx \sin \omega t$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \omega^2 x^2) - mgx \sin \omega t, \quad q_1 = x, \quad \dot{q}_1 = \dot{x}$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L} \Rightarrow m\ddot{x} = m\omega^2 x - mg \sin \omega t$$

مکانیک لاگرانژی

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \mathcal{L} \right] = \frac{\partial}{\partial x} \mathcal{L} \Rightarrow m\ddot{x} = m\omega^2 x - mg \sin \omega t$$

$$\ddot{x} - \omega^2 x = -g \sin \omega t$$

$$\ddot{x}_h - \omega^2 x_h = 0 : \quad x_h(t) = Ae^{\omega t} + Be^{-\omega t}$$

$$\ddot{x}_n - \omega^2 x_n = -g \sin \omega t : \quad \begin{cases} x_n(t) = C_1 \sin \omega t + C_2 \cos \omega t \\ \dot{x}_n(t) = \omega C_1 \cos \omega t - \omega C_2 \sin \omega t \\ \ddot{x}_n(t) = -\omega^2 C_1 \sin \omega t - \omega^2 C_2 \cos \omega t \end{cases}$$

$$-2\omega^2 C_1 \sin \omega t - 2\omega^2 C_2 \cos \omega t = -g \sin \omega t$$

مکانیک لاگرانژی

$$-2\omega^2 C_1 \sin \omega t - 2\omega^2 C_2 \cos \omega t = -g \sin \omega t$$

$$C_1 = -\frac{g}{2\omega^2}, \quad C_2 = 0$$

$$x_n(t) = -\frac{g}{2\omega^2} \sin \omega t$$

$$x(t) = x_h(t) + x_n(t)$$

$$x(t) = Ae^{\omega t} + Be^{-\omega t} - \frac{g}{2\omega^2} \sin \omega t$$