جلسه سیزدهم

مکانیک تحلیلی

محمدرضا مظفری
گروه فیزیک، دانشکده علوم پایه
دانشگاه قم
اسفند ۹۸

یادآوری

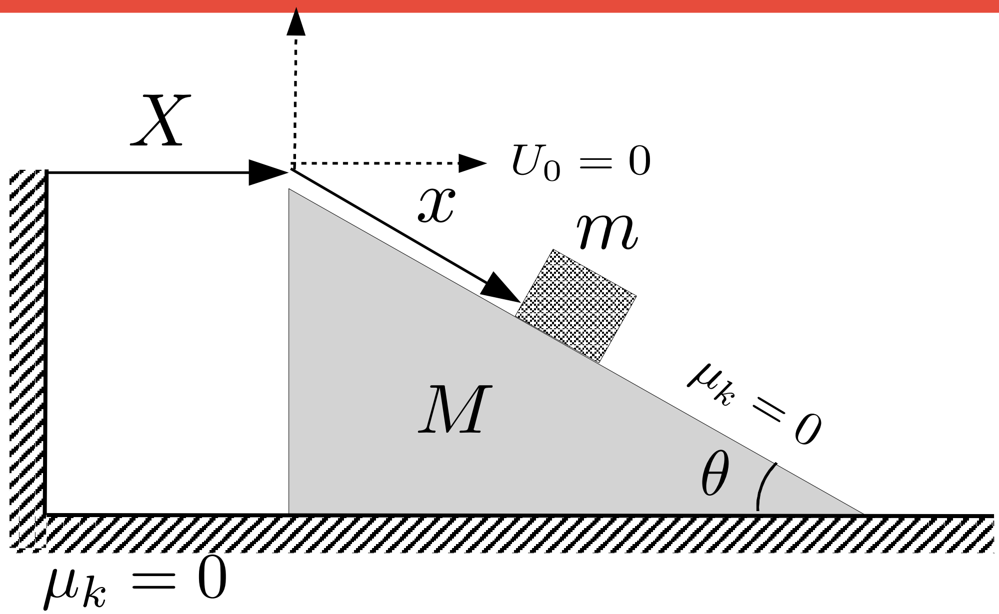
انرژی جنبش کل :
$$T = \sum_{i}^{N} \left(\frac{1}{2} m_i \dot{x}_i^2 + \frac{1}{2} m_i \dot{y}_i^2 + \frac{1}{2} m_i \dot{z}_i^2 \right)$$

انرژی پتانسیل کل :
$$V = V(q_1, q_2, \cdots, q_k, \cdots, q_n)$$

$$\mathcal{L} = T - V$$

معادلات لاگرانژ:
$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial}{\partial \dot{q}_k} \mathcal{L} \right] = \frac{\partial}{\partial q_k} \mathcal{L}, \quad 1 \le k \le n$$

مکانیک لاگرانژی



$$m : (X + x\cos\theta, -x\sin\theta)$$

$$m : (\dot{X} + \dot{x}\cos\theta, -\dot{x}\sin\theta)$$

$$M : (X, 0)$$

$$M : (\dot{X}, 0)$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + \dot{x}^2 + 2\dot{X}\dot{x}\cos\theta), \quad V = -mgx\sin\theta$$

$$\mathcal{L} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + \dot{x}^2 + 2\dot{X}\dot{x}\cos\theta) + mgx\sin\theta$$

$$q_1 = x, \dot{q}_1 = \dot{x}, \quad q_2 = X, \dot{q}_2 = \dot{X}$$

مکانیک لاگرانژی

$$\mathcal{L} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + \dot{x}^2 + 2\dot{X}\dot{x}\cos\theta) + mgx\sin\theta$$

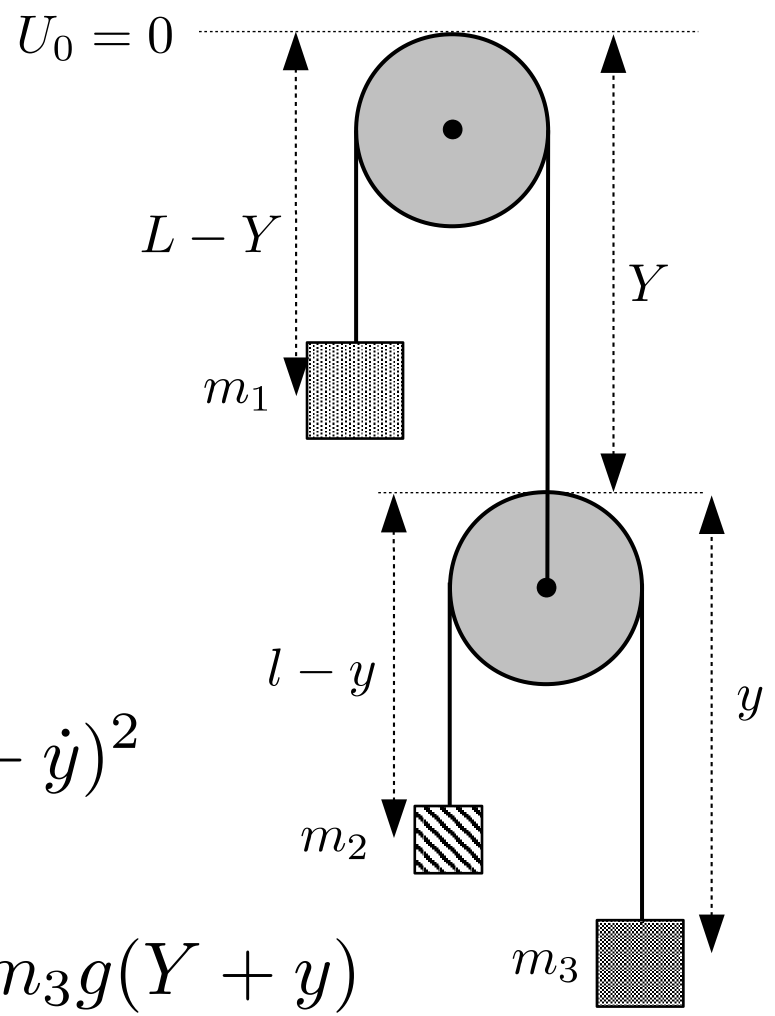
$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial\mathcal{L}}{\partial\dot{x}}\right] = \dfrac{\partial\mathcal{L}}{\partial x} \Rightarrow m(\ddot{x} + \ddot{X}\cos\theta) = mg\sin\theta \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial\mathcal{L}}{\partial\dot{X}}\right] = \dfrac{\partial\mathcal{L}}{\partial X} \Rightarrow (M+m)\ddot{X} + m\ddot{x}\cos\theta = 0 \end{cases}$$

$$\begin{cases} \ddot{x} + \ddot{X}\cos\theta = g\sin\theta \\ (M+m)\ddot{X} + m\ddot{x}\cos\theta = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = \dfrac{(M+m)g\sin\theta}{M+m-m\cos^2\theta} \\ \ddot{X} = -\dfrac{mg\sin\theta\cos\theta}{M+m-m\cos^2\theta} \end{cases}$$

$$m_1 : L - Y, \quad -\dot{Y}$$

$$m_2 : Y + l - y, \quad \dot{Y} - \dot{y}$$

$$m_3 : Y + y, \quad \dot{Y} + \dot{y}$$



$$q_1 = Y, \quad \dot{q}_1 = \dot{Y}, \quad q_2 = y, \quad \dot{q}_2 = \dot{y}$$

$$T = \frac{1}{2} m_1 \dot{Y}^2 + \frac{1}{2} m_2 (\dot{Y} - \dot{y})^2 + \frac{1}{2} m_3 (\dot{Y} + \dot{y})^2$$

$$V = -m_1 g(L - Y) - m2g(Y + l - y) - m_3 g(Y + y)$$

مکانیک لاگرانژی

$$q_1 = Y, \quad \dot{q}_1 = \dot{Y}, \quad q_2 = y, \quad \dot{q}_2 = \dot{y}$$

$$T = \frac{1}{2}m_1\dot{Y}^2 + \frac{1}{2}m_2(\dot{Y} - \dot{y})^2 + \frac{1}{2}m_3(\dot{Y} + \dot{y})^2$$

$$V = -m_1 g(L - Y) - m2g(Y + l - y) - m_3 g(Y + y)$$

$$\mathcal{L} = \frac{1}{2}m_1\dot{Y}^2 + \frac{1}{2}m_2(\dot{Y} - \dot{y})^2 + \frac{1}{2}m_3(\dot{Y} + \dot{y})^2$$
$$+m_1 g(L - Y) + m2g(Y + l - y) + m_3 g(Y + y)$$

مکانیک لاگرانژی

$$\mathcal{L} = \frac{1}{2}m_1\dot{Y}^2 + \frac{1}{2}m_2(\dot{Y} - \dot{y})^2 + \frac{1}{2}m_3(\dot{Y} + \dot{y})^2$$
$$+m_1 g(L - Y) + m_2 g(Y + l - y) + m_3 g(Y + y)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{Y}}\right] = \frac{\partial \mathcal{L}}{\partial Y}$$

$$m_1\ddot{Y} + m_2(\ddot{Y} - \ddot{y}) + m_3(\ddot{Y} + \ddot{y}) = -m_1 g + m_2 g + m_3 g$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\right] = \frac{\partial \mathcal{L}}{\partial y}$$

$$-m_2(\ddot{Y} - \ddot{y}) + m_3(\ddot{Y} + \ddot{y}) = -m_2 g + m_3 g$$

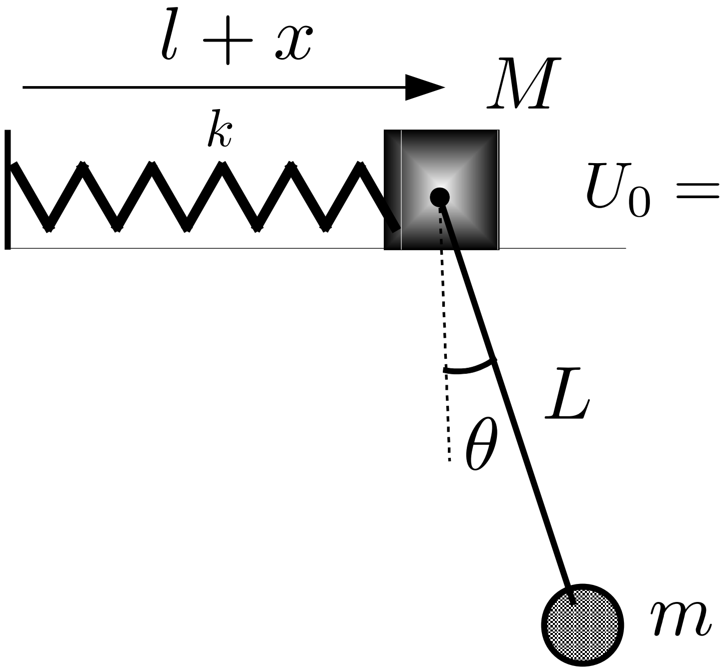
مکانیک لاگرانژی

$$\begin{cases} m_1\ddot{Y} + m_2(\ddot{Y} - \ddot{y}) + m_3(\ddot{Y} + \ddot{y}) = -m_1 g + m_2 g + m_3 g \\ -m_2(\ddot{Y} - \ddot{y}) + m_3(\ddot{Y} + \ddot{y}) = -m_2 g + m_3 g \end{cases}$$

$$\boxed{\begin{cases} (m_1 + m_2 + m_3)\ddot{Y} + (-m_2 + m_3)\ddot{y} = (-m_1 + m_2 + m_3)g \\ (-m_2 + m_3)\ddot{Y} + (m_2 + m_3)\ddot{y} = (-m_2 + m_3)g \end{cases}}$$

اگر $m_1 = m,\ m_2 = 2m,\ m_3 = 3m$

$$\begin{cases} 6\ddot{Y} + \ddot{y} = 4g \\ \ddot{Y} + 5\ddot{y} = g \end{cases} \Rightarrow \ddot{Y} = \frac{19}{29}g, \quad \ddot{y} = \frac{2}{29}g$$



$$m : (l + x + L\sin\theta, -L\cos\theta)$$

$$m : (\dot{x} + L\dot{\theta}\cos\theta, L\dot{\theta}\sin\theta)$$

$$M : (l + x, 0)$$

$$M : (\dot{x}, 0)$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad q_2 = \theta, \quad \dot{q}_2 = \dot{\theta}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos\theta)$$

$$V = \frac{1}{2}kx^2 - mgL\cos\theta$$

مکانیک لاگرانژی

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos\theta)$$

$$V = \frac{1}{2}kx^2 - mgL\cos\theta$$

$$\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos\theta) - \frac{1}{2}kx^2 + mgL\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] = \frac{\partial\mathcal{L}}{\partial x}$$

$$(M+m)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = -kx$$

محمدرضا مظفری
گروه فیزیک، دانشکده علوم پایه، دانشگاه قم

مکانیک لاگرانژی

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos\theta)$$

$$V = \frac{1}{2}kx^2 - mgL\cos\theta$$

$$\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}\cos\theta) - \frac{1}{2}kx^2 + mgL\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right] = \frac{\partial\mathcal{L}}{\partial\theta}$$

$$mL^2\ddot{\theta} + mL\ddot{x}\cos\theta - \cancel{mL\dot{x}\dot{\theta}\sin\theta} = \cancel{-mL\dot{x}\dot{\theta}\sin\theta} - mgL\sin\theta$$

$$mL^2\ddot{\theta} + mL\ddot{x}\cos\theta = -mgL\sin\theta$$

مکانیک لاگرانژی

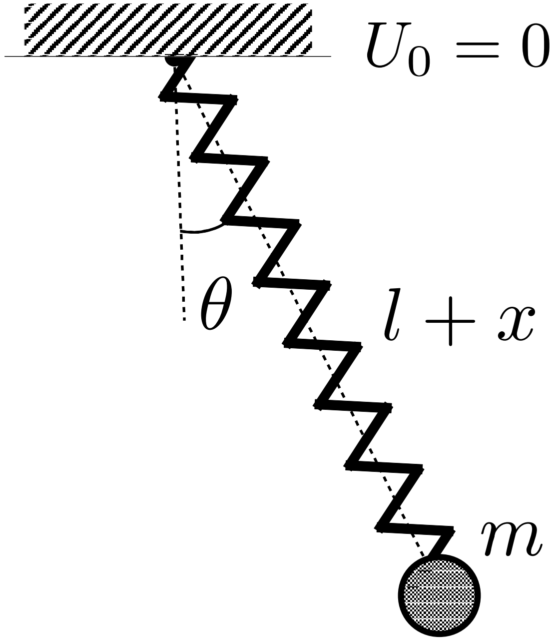
$$\begin{cases} (M+m)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = -kx \\ mL^2\ddot{\theta} + mL\ddot{x}\cos\theta = -mgL\sin\theta \end{cases}$$

$$\theta \to 0$$

$$\begin{cases} mL\ddot{\theta}\cos\theta \to mL\ddot{\theta}, \\ mL\ddot{x}\cos\theta \to mL\ddot{x}, \end{cases} \qquad \begin{array}{c} mL\dot{\theta}^2\sin\theta \to mL\dot{\theta}^2\theta \to 0 \\ mgL\sin\theta \to mgL\theta \end{array}$$

$$\div mL \; \Rightarrow \; \begin{cases} (M+m)\ddot{x} + mL\ddot{\theta} = -kx \\ L\ddot{\theta} + \ddot{x} = -g\theta \end{cases}$$

مکانیک لاگرانژی



$$m : ((l + x)\sin\theta, -(l + x)\cos\theta)$$

$$m : (\dot{x}\sin\theta + (l + x)\dot{\theta}\cos\theta,$$
$$-\dot{x}\cos\theta + (l + x)\dot{\theta}\sin\theta)$$

$$q_1 = x, \quad \dot{q}_1 = \dot{x}, \quad q_2 = \theta, \quad \dot{q}_2 = \dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + (l + x)^2\dot{\theta}^2)$$

$$V = \frac{1}{2}kx^2 - mg(l + x)\cos\theta$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + (l + x)^2\dot{\theta}^2) - \frac{1}{2}kx^2 + mg(l + x)\cos\theta$$

مکانیک لاگرانژی

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\dot{\theta}^2) - \frac{1}{2}kx^2 + mg(l+x)\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] = \frac{\partial \mathcal{L}}{\partial x}$$

$$m\ddot{x} = m(l+x)\dot{\theta}^2 - kx + mg\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} = -mg(l+x)\sin\theta$$

$$\begin{cases} m\ddot{x} = m(l+x)\dot{\theta}^2 - kx + mg\cos\theta \\ m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} = -mg(l+x)\sin\theta \end{cases}$$

$$\theta \to 0$$

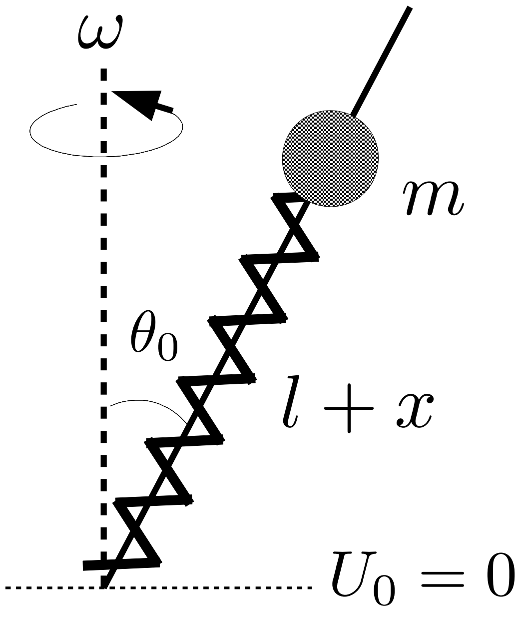
$$mg\cos\theta \to mg, \qquad mg(l+x)\sin\theta \to mg(l+x)\theta$$

$\div m$

$\div m(l+x)$

$$\begin{cases} \ddot{x} = (l+x)\dot{\theta}^2 - \dfrac{k}{m}x + g \\ (l+x)\ddot{\theta} + 2\dot{x}\dot{\theta} = -g\theta \end{cases}$$

مکانیک لاگرانژی



$$m:((l+x)\sin\theta_0\cos\phi,$$
$$(l+x)\sin\theta_0\sin\phi,$$
$$(l+x)\cos\theta_0)$$

$$m:(\dot{x}\sin\theta_0\cos\phi-(l+x)\dot{\phi}\sin\theta_0\sin\phi,$$
$$\dot{x}\sin\theta_0\sin\phi+(l+x)\dot{\phi}\sin\theta_0\cos\phi,$$
$$\dot{x}\cos\theta_0)$$

$$\dot{\phi}=\omega$$

$$m:(\dot{x}\sin\theta_0\cos\phi-(l+x)\omega\sin\theta_0\sin\phi,$$
$$\dot{x}\sin\theta_0\sin\phi+(l+x)\omega\sin\theta_0\cos\phi,\ \dot{x}\cos\theta_0)$$

مکانیک لاگرانژی

$$m : (\dot{x}\sin\theta_0\cos\phi - (l+x)\omega\sin\theta_0\sin\phi,$$

$$\dot{x}\sin\theta_0\sin\phi + (l+x)\omega\sin\theta_0\cos\phi, \quad \dot{x}\cos\theta_0)$$

$$q_1 = x, \qquad \dot{q}_1 = \dot{x}$$

$$T = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\omega^2\sin^2\theta_0), \quad V = \frac{1}{2}kx^2 + mg(l+x)\cos\theta_0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\omega^2\sin^2\theta_0) - \frac{1}{2}kx^2 - mg(l+x)\cos\theta_0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow m\ddot{x} = m(l+x)\omega^2\sin^2\theta_0 - kx - mg\cos\theta_0$$

مکانیک لاگرانژی

$$m\ddot{x} = m(l+x)\omega^2 \sin^2\theta_0 - kx - mg\cos\theta_0$$

$\div m$ →

$$\ddot{x} = (l+x)\omega^2 \sin^2\theta_0 - \frac{k}{m}x - g\cos\theta_0$$

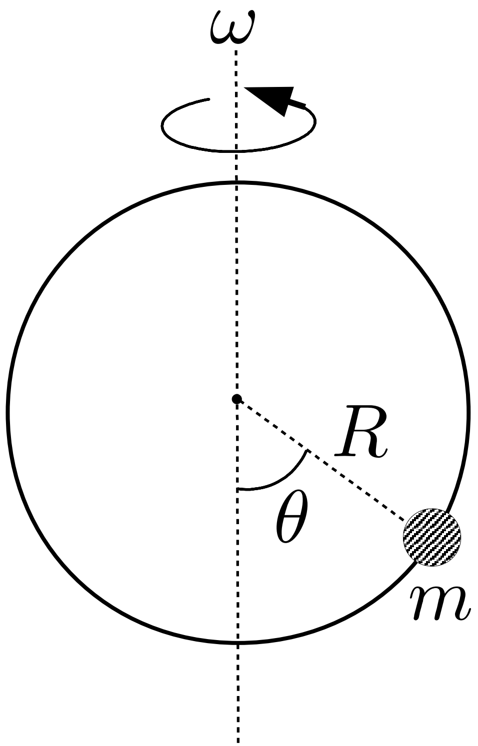
$\omega_0^2 = \dfrac{k}{m}$ →

$$\ddot{x} = (l+x)\omega^2 \sin^2\theta_0 - \omega_0^2 x - g\cos\theta_0$$

$$\boxed{\ddot{x} + (\omega_0^2 - \omega^2\sin^2\theta_0)x = l\omega^2\sin^2\theta_0 - g\cos\theta_0}$$

فرکانس نوسانات ذره $= \sqrt{\omega_0^2 - \omega^2\sin^2\theta_0}$

مکانیک لاگرانژی



$$m:(R\sin\theta\cos\phi, R\sin\theta\sin\phi, -R\cos\theta)$$

$$m:(R\dot\theta\cos\theta\cos\phi - R\dot\phi\sin\theta\sin\phi,$$
$$R\dot\theta\cos\theta\sin\phi + R\dot\phi\sin\theta\cos\phi,$$
$$-R\dot\theta\sin\theta)$$

$\dot\phi = \omega$

$$m:(R\dot\theta\cos\theta\cos\phi - R\omega\sin\theta\sin\phi,$$
$$R\dot\theta\cos\theta\sin\phi + R\omega\sin\theta\cos\phi,$$
$$-R\dot\theta\sin\theta)$$

مکانیک لاگرانژی

$$m : (R\dot{\theta}\cos\theta\cos\phi - R\omega\sin\theta\sin\phi,$$
$$R\dot{\theta}\cos\theta\sin\phi + R\omega\sin\theta\cos\phi, -R\dot{\theta}\sin\theta)$$

$$q_1 = \theta, \quad \dot{q}_1 = \dot{\theta}$$

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta), \quad V = -mgR\cos\theta$$

$$\mathcal{L} = \frac{1}{2}m(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta) + mgR\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{x}}\right] = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

مکانیک لاگرانژی

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta$$

$\div mR^2$ $\Rightarrow$

$$\ddot{\theta} = \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

$\omega_0^2 = \dfrac{g}{R}$ $\Rightarrow$

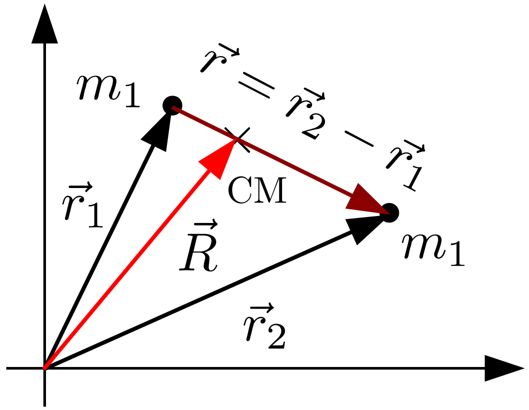
$$\ddot{\theta} = \omega^2 \sin\theta\cos\theta - \omega_0^2\sin\theta$$

$$\theta \to 0$$

$$\omega^2 \sin\theta\cos\theta \to \omega^2\theta, \qquad \frac{g}{R}\sin\theta \to \frac{g}{R}\theta$$

$$\ddot{\theta} = \omega^2\theta - \omega_0^2\theta \Rightarrow \boxed{\ddot{\theta} + (\omega_0^2 - \omega^2)\theta = 0}$$ فرکانس نوسانات ذره $= \sqrt{\omega_0^2 - \omega^2}$

مکانیک لاگرانژی



نیروهای مرکزی

$$T=\frac{1}{2}m_1\dot{\vec{r}}_1\cdot\dot{\vec{r}}_1+\frac{1}{2}m_2\dot{\vec{r}}_2\cdot\dot{\vec{r}}_2$$

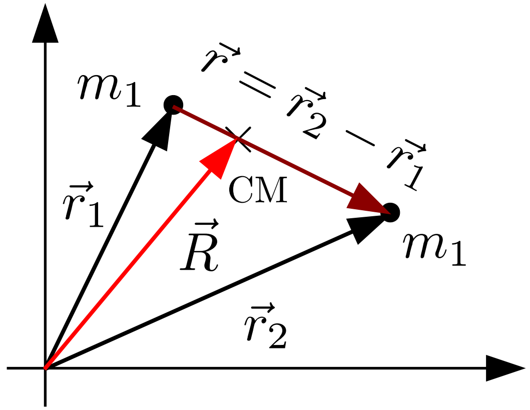
$$V=V(|\vec{r}_2-\vec{r}_1|),$$ مثال: $$V=\frac{k}{|\vec{r}_2-\vec{r}_1|}$$

$$\begin{cases} m_1\vec{r}_1+m_2\vec{r}_2=(m_1+m_2)\vec{R} \\ \vec{r}_2-\vec{r}_1=\vec{r} \end{cases} \Rightarrow \begin{cases} \vec{r}_1=-\frac{\mu}{m_1}\vec{r}+\vec{R} \\ \vec{r}_2=\frac{\mu}{m_2}\vec{r}+\vec{R} \end{cases}$$

$$\begin{cases} \dot{\vec{r}}_1=-\frac{\mu}{m_1}\dot{\vec{r}}+\dot{\vec{R}} \\ \dot{\vec{r}}_2=\frac{\mu}{m_2}\dot{\vec{r}}+\dot{\vec{R}} \end{cases} \Rightarrow T=\frac{1}{2}\mu\dot{\vec{r}}\cdot\dot{\vec{r}}+\frac{1}{2}(m_1+m_2)\dot{\vec{R}}\cdot\dot{\vec{R}},\ V=V(|\vec{r}|)$$

مکانیک لاگرانژی

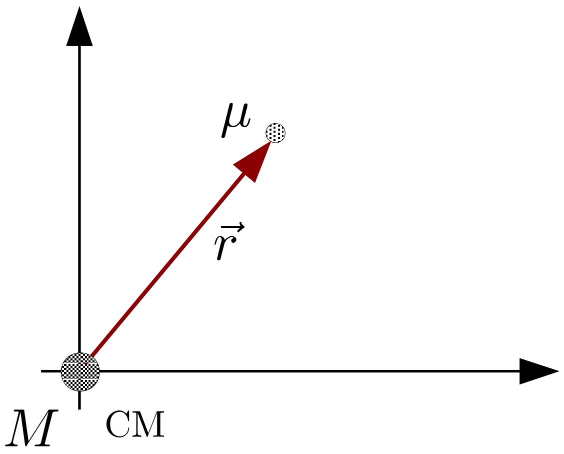
نیروهای مرکزی



$$M = m_1 + m_2$$

$$\dot{\vec{r}} \cdot \dot{\vec{r}} = \dot{r}^2, \qquad \dot{\vec{R}} \cdot \dot{\vec{R}} = \dot{R}^2$$

$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}M\dot{R}^2 - V(r)$$



$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 - V(r)$$

$$\vec{r} = r\hat{e}_r, \quad \hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

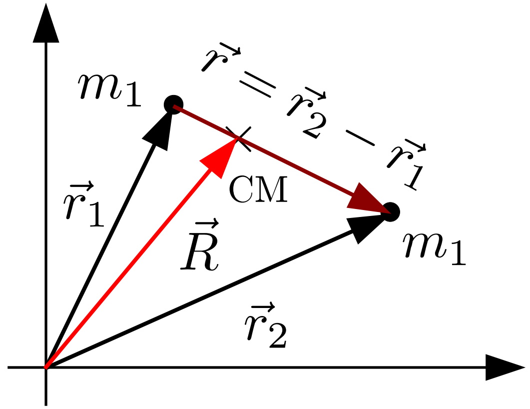
$$q_1 = r,\ q_2 = \theta$$

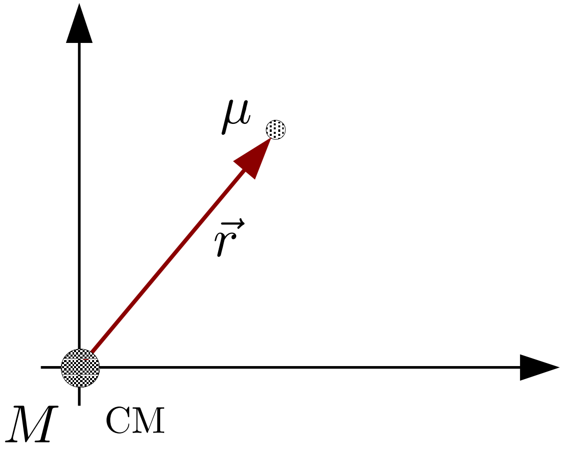
فرض: $\vec{R} = 0$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta, \quad \hat{e}_r = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

مکانیک لاگرانژی

نیروهای مرکزی





فرض: $\vec{R} = 0$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}\cdot\dot{\vec{r}} - V(r)$$

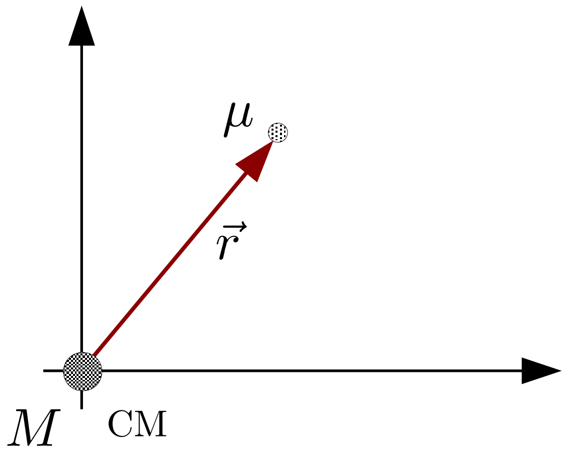
$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$q_1 = r,\ q_2 = \theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{r}}\mathcal{L}\right] = \frac{\partial}{\partial r}\mathcal{L} \Rightarrow \mu\ddot{r} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r} + \mu r\dot{\theta}^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{\theta}}\mathcal{L}\right] = \frac{\partial}{\partial \theta}\mathcal{L} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\mu r^2\dot{\theta}) = 0$$

نیروهای مرکزی



فرض: $\vec{R} = 0$

$$q_1 = r,\ q_2 = \theta$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

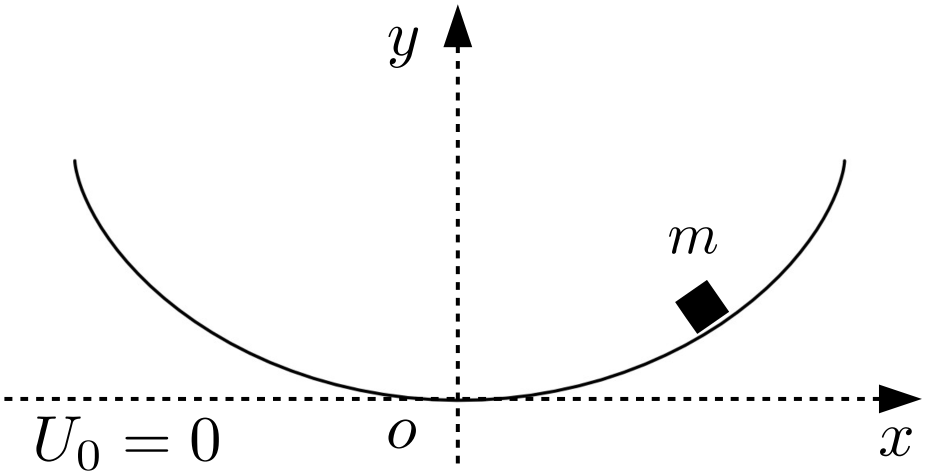
$$\mu\ddot{r} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r} + \mu r\dot{\theta}^2, \quad f = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}$$

$$\boxed{\mu\ddot{r} = f(r) + \mu r\dot{\theta}^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu r^2\dot{\theta}) = 0 \Rightarrow \boxed{l = \mu r^2\dot{\theta}}$$

$$\Rightarrow \boxed{\mu\ddot{r} = f(r) + \frac{l^2}{\mu r^3}}$$

مکانیک لاگرانژی



$$\begin{cases} x = A(2\phi + \sin 2\phi) \\ y = A(1 - \cos 2\phi) \end{cases} \qquad A = \frac{a}{2}$$

$$0 \leq \phi \leq \pi$$

$$q_1 = \phi, \quad \dot{q}_1 = \dot{\phi}$$

$$\dot{x} = 2A\dot{\phi}(1 + \cos 2\phi), \quad \dot{y} = 2A\dot{\phi}\sin 2\phi$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = 4mA^2\dot{\phi}^2(1 + \cos 2\phi)$$

$$\mathcal{L} = T - V = 4mA^2\dot{\phi}^2(1 + \cos 2\phi) - mgA(1 - \cos 2\phi)$$

مکانیک لاگرانژی

$$\mathcal{L} = 4mA^2\dot{\phi}^2(1+\cos 2\phi) - mgA(1-\cos 2\phi)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right] = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[8mA^2\dot{\phi}(1+\cos 2\phi)\right] = -8mA^2\dot{\phi}^2\sin 2\phi - 2mgA\sin 2\phi$$

$$8mA^2\ddot{\phi}(1+\cos 2\phi) - 16mA^2\dot{\phi}^2\sin 2\phi =$$

$$-8mA^2\dot{\phi}^2\sin 2\phi - 2mgA\sin 2\phi$$

$$8mA^2\ddot{\phi}(1+\cos 2\phi) = 8mA^2\dot{\phi}^2\sin 2\phi - 2mgA\sin 2\phi$$

مکانیک لاگرانژی

$$8mA^2\ddot{\phi}(1+\cos 2\phi) = 8mA^2\dot{\phi}^2\sin 2\phi - 2mgA\sin 2\phi$$

$\div 2mA$ →

$$4A\ddot{\phi}(1+\cos 2\phi) = 4A\dot{\phi}^2\sin 2\phi - g\sin 2\phi$$

$$4A\ddot{\phi}2\cos^2\phi = 4A\dot{\phi}^2 2\sin\phi\cos\phi - g2\sin\phi\cos\phi$$

$\div 2\cos\phi$ →

$$4A\ddot{\phi}\cos\phi = 4A\dot{\phi}^2\sin\phi - g\sin\phi$$

$$\begin{cases} 4A(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi) = -g\sin\phi \\ \dfrac{d^2}{dt^2}\sin\phi = \ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi \end{cases} \Rightarrow 4A\frac{d^2}{dt^2}\sin\phi = -g\sin\phi$$

مکانیک لاگرانژی

$$4A\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin\phi = -g\sin\phi$$

$\div 4A$ →

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin\phi + \frac{g}{4A}\sin\phi = 0$$

$\eta = \sin\phi$ →

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta + \frac{g}{4A}\eta = 0$$

$$\omega_0 = \sqrt{\frac{g}{4A}} = \sqrt{\frac{g}{2a}}$$