جلسه شانزدهم

مکانیک تحلیلی

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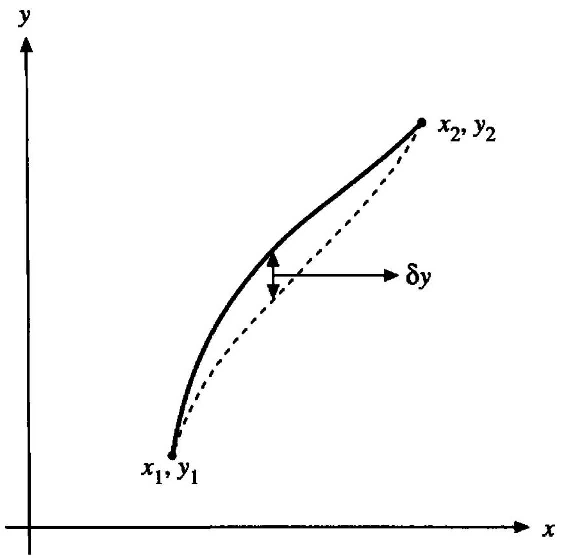
اسفند ۹۸

مکانیک لاگرانژی

$$\mathcal{J} = \int_{x_1}^{x_2} f(x, y, \dot{y})\mathrm{d}x$$

* در روش وردشی کمیتی که باید کمینه(یا بیشنیه) شود بصورت یک انتگرال ظاهر می‌شود.

* تابع f کمیتی معلوم از متغییرهای $x, y(x), \dot{y}(x) = \mathrm{d}y/\mathrm{d}x$ است ولی بستگی y به x تعیین نشده است یا $y(x)$ **مجهول** است.



* اگر چه حدود انتگرالگیری x_1 و x_2 معلوم است ولی مسیر انتگرالی گیری در صفحه‌ی xy معلوم نیست.

$$y_1 = (x_1), \quad y_2 = y(x_2)$$

مکانیک لاگرانژی

$$\mathcal{J} = \int_{x_1}^{x_2} f(x, y, \dot{y})\mathrm{d}x$$

$$y(x) \to y(x, \alpha) = y(x) + \alpha\eta(x)$$

$$y_1 = (x_1), \quad y_2 = y(x_2) \Rightarrow \eta(x_1) = \eta(x_2) = 0$$

$$\mathcal{J}(\alpha) = \int_{x_1}^{x_2} f(x, y(x, \alpha), \dot{y}(x, \alpha))\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}}{\partial \alpha}\right]_{\alpha=0} = 0$$

متغیر

$$\mathcal{J}(\alpha) = \int_{x_1}^{x_2} f(x, y(x,\alpha), \dot{y}(x,\alpha))\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}}{\partial \alpha}\right]_{\alpha=0} = 0$$

$$\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y(x,\alpha)}\frac{\partial y(x,\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}(x,\alpha)}\frac{\partial \dot{y}(x,\alpha)}{\partial \alpha}\right]\mathrm{d}x$$

$$y(x,\alpha) = y(x) + \alpha\eta(x) : \qquad \frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x)$$

$$\dot{y}(x,\alpha) = \dot{y}(x) + \alpha\frac{\mathrm{d}\eta(x)}{\mathrm{d}x} : \qquad \frac{\partial \dot{y}(x,\alpha)}{\partial \alpha} = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x}$$

مکانیک لاگرانژی

$$\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y(x,\alpha)} \frac{\partial y(x,\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}(x,\alpha)} \frac{\partial \dot{y}(x,\alpha)}{\partial \alpha} \right] \mathrm{d}x$$

نکته

$$y(x,\alpha) = y(x) + \alpha \eta(x) : \quad \frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x)$$

$$\dot{y}(x,\alpha) = \dot{y}(x) + \alpha \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} : \quad \frac{\partial \dot{y}(x,\alpha)}{\partial \alpha} = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x}$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} \right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial \dot{y}} \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} \right] \mathrm{d}x$$

$$\frac{\partial f}{\partial \dot{y}} \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\partial f}{\partial \dot{y}} \eta(x) \right] - \eta(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial \dot{y}} \right)$$

مکانیک لاگرانژی

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial \dot{y}}\frac{\mathrm{d}\eta(x)}{\mathrm{d}x}\right]\mathrm{d}x$$

$$\frac{\partial f}{\partial \dot{y}}\frac{\mathrm{d}\eta(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial f}{\partial \dot{y}}\eta(x)\right] - \eta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta(x)\mathrm{d}x + \int_{x_1}^{x_2} \mathrm{d}\left[\frac{\partial f}{\partial \dot{y}}\eta(x)\right]$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta(x)\mathrm{d}x + \left[\frac{\partial f}{\partial \dot{y}}\eta(x)\right]_{x_1}^{x_2}$$

مکانیک لاگرانژی

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \eta(x)\mathrm{d}x + \left[\frac{\partial f}{\partial \dot{y}}\eta(x)\right]_{x_1}^{x_2}$$

تک متغیر

$$\eta(x_1) = \eta(x_2) = 0 \Rightarrow \left[\frac{\partial f}{\partial \dot{y}}\eta(x)\right]_{x_1}^{x_2} = 0$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \eta(x)\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0 \Rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$ معادله اویلر

مکانیک لاگرانژی

چند متغیر

$$\mathcal{J}=\int_{x_1}^{x_2} f(x,\{y\},\{\dot{y}\})\mathrm{d}x,\qquad \dot{y}_i=\frac{\partial y_i}{\partial x}$$

$$y_i(x)\to y_i(x,\alpha)=y_i(x)+\alpha\eta_i(x),\qquad i=1,2,\cdots,n$$

$$y_i^{(1)}=y_i(x_1),\qquad y_i^{(2)}=y_i(x_2)\Rightarrow \eta_i(x_1)=\eta_i(x_2)=0$$

$$\mathcal{J}(\alpha)=\int_{x_1}^{x_2} f(x,\{y(x,\alpha)\},\{\dot{y}(x,\alpha)\})\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial\alpha}\right]_{\alpha=0}=0$$

مکانیک لاگرانژی

چند متغیر

$$\mathcal{J}(\alpha) = \int_{x_1}^{x_2} f(x, \{y(x,\alpha)\}, \{\dot{y}(x,\alpha)\})\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$$

$$\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i(x,\alpha)} \frac{\partial y_i(x,\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_i(x,\alpha)} \frac{\partial \dot{y}_i(x,\alpha)}{\partial \alpha}\right] \mathrm{d}x$$

$$y_i(x,\alpha) = y_i(x) + \alpha \eta_i(x) : \quad \frac{\partial y_i(x,\alpha)}{\partial \alpha} = \eta_i(x)$$

$$\dot{y}_i(x,\alpha) = \dot{y}_i(x) + \alpha \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x} : \quad \frac{\partial \dot{y}_i(x,\alpha)}{\partial \alpha} = \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x}$$

$$\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i(x,\alpha)} \frac{\partial y_i(x,\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_i(x,\alpha)} \frac{\partial \dot{y}_i(x,\alpha)}{\partial \alpha} \right] \mathrm{d}x$$

چند متغیر

$$y_i(x,\alpha) = y_i(x) + \alpha \eta_i(x) : \qquad \frac{\partial y_i(x,\alpha)}{\partial \alpha} = \eta_i(x)$$

$$\dot{y}_i(x,\alpha) = \dot{y}_i(x) + \alpha \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x} : \qquad \frac{\partial \dot{y}_i(x,\alpha)}{\partial \alpha} = \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x}$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha} \right]_{\alpha=0} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} \eta_i(x) + \frac{\partial f}{\partial \dot{y}_i} \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x} \right] \mathrm{d}x$$

$$\frac{\partial f}{\partial \dot{y}_i} \frac{\mathrm{d}\eta_i(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\partial f}{\partial \dot{y}_i} \eta_i(x) \right] - \eta_i(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial \dot{y}_i} \right)$$

مکانیک لاگرانژی

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}_i}\right)\right] \eta_i(x)\mathrm{d}x + \sum_i \left[\frac{\partial f}{\partial \dot{y}_i}\eta_i(x)\right]_{x_1}^{x_2}$$

چند متغیر

$$\eta_i(x_1) = \eta_i(x_2) = 0 \Rightarrow \left[\frac{\partial f}{\partial \dot{y}_i}\eta_i(x)\right]_{x_1}^{x_2} = 0$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}_i}\right)\right] \eta_i(x)\mathrm{d}x$$

$$\left[\frac{\partial \mathcal{J}(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0 \Rightarrow \boxed{\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0}$$

معادله اویلر

$$i = 1, 2, \cdots, n$$

مکانیک لاگرانژی

چند متغیر

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0, \quad i = 1, 2, \cdots, n$$

$$f \to \mathcal{L} = T - V$$

$$x \to t$$

$$\{y\} \to \{q\}$$

$$\{\dot{y}\} \to \{\dot{q}\}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \cdots, n$$

معادله اویلر-لاگرانژ

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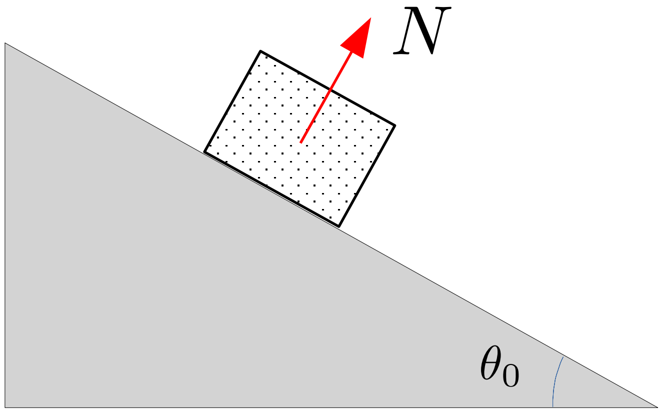
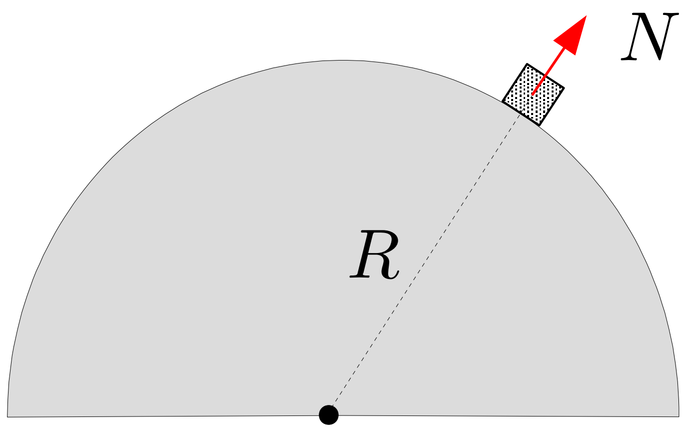
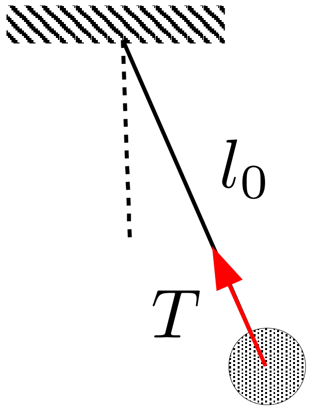
مکانیک لاگرانژی



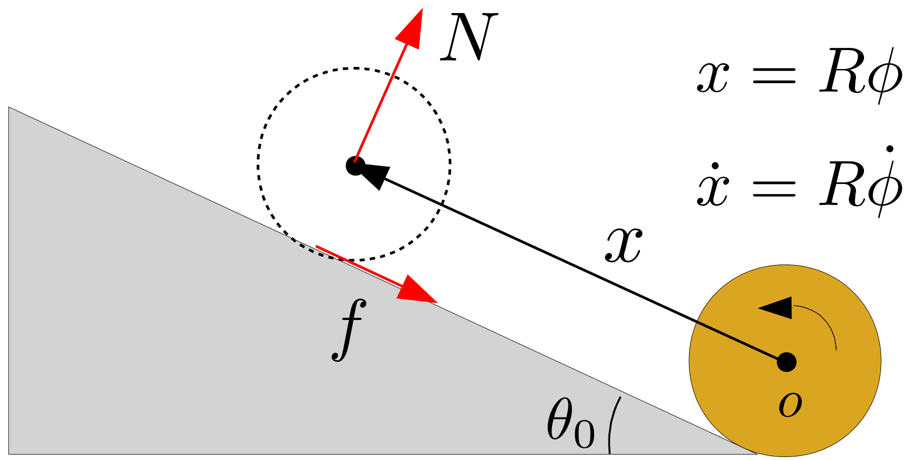
$\theta = \theta_0$



$r = R$

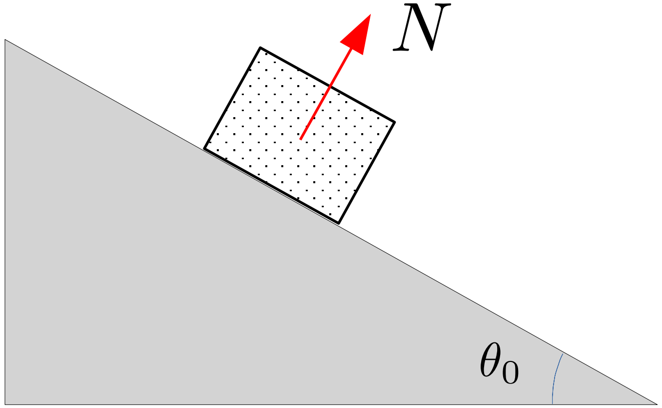


$r = l_0$

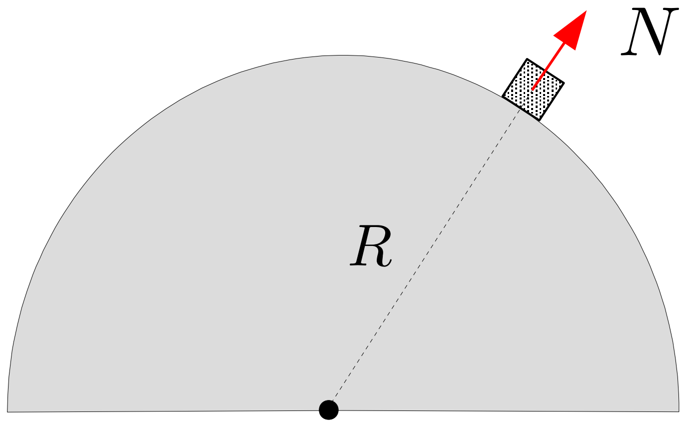


$\theta = \theta_0$

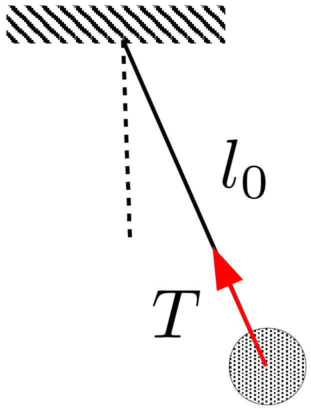
مکانیک لاگرانژی



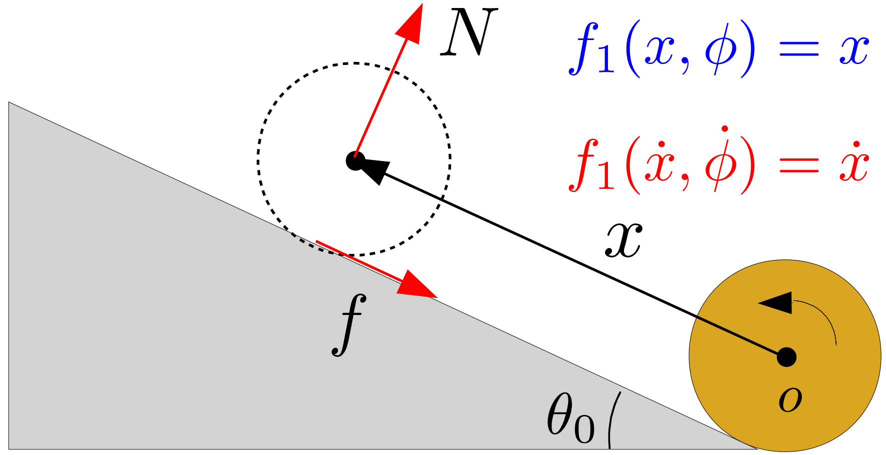
$$f(\theta) = \theta - \theta_0$$



$$f(r) = r - R$$



$$f(r) = r - l_0$$



$$f_1(x, \phi) = x - R\phi$$

$$f_1(\dot{x}, \dot{\phi}) = \dot{x} - R\dot{\phi}$$

$$f_2(\theta) = \theta - \theta_0$$

مکانیک لاگرانژی

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \cdots, n$$

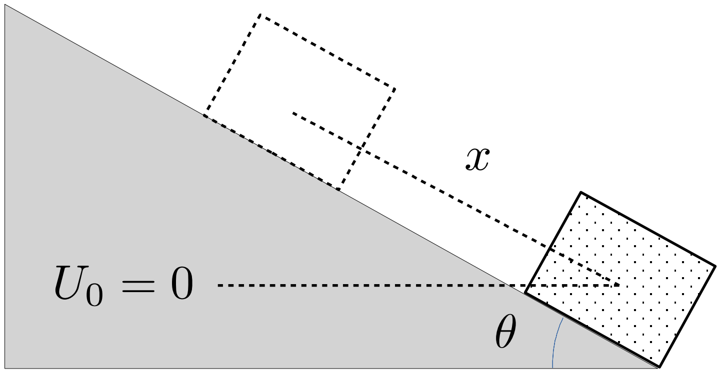
قیدها هولونومیک $f_k = f_k(\{q\}), \quad k = 1, 2, \cdots, M$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_k \lambda_k \frac{\partial f_k}{\partial q_i}, \quad i = 1, 2, \cdots, n}$$

قیدها غیر هولونومیک $f_k = f_k(\{\dot{q}\}), \quad k = 1, 2, \cdots, M$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} + \sum_k \lambda_k \frac{\partial f_k}{\partial \dot{q}_i}, \quad i = 1, 2, \cdots, n}$$

مکانیک لاگرانژی



$$T = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2), \quad V = mgx\sin\theta$$

$$f(\theta) = \theta - \theta_0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) - mgx\sin\theta$$

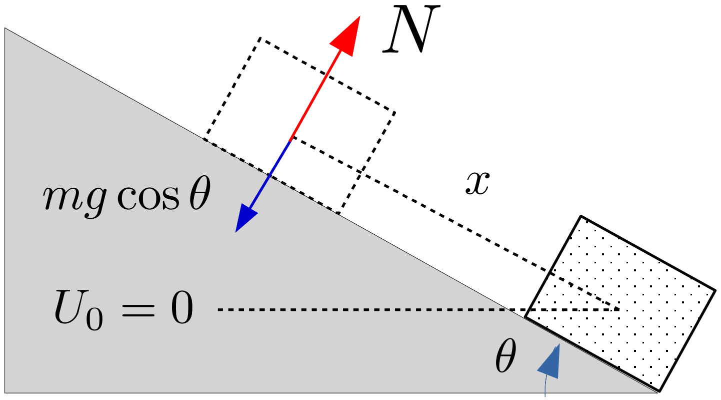
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta}$$

$$m\ddot{x} = mx\dot{\theta}^2 - mg\sin\theta \qquad mx^2\ddot{\theta} + 2mx\dot{x}\dot{\theta} = -mgx\cos\theta + \lambda$$

$$f(\theta) = \theta - \theta_0 = 0 \Rightarrow \theta = \theta_0 \Rightarrow \ddot{\theta} = \dot{\theta} = 0$$

$$\ddot{x} = -g\sin\theta_0 \qquad 0 = -mgx\cos\theta_0 + \lambda$$

مکانیک لاگرانژی



$$T = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2), \quad V = mgx\sin\theta$$

$$f(\theta) = \theta - \theta_0$$

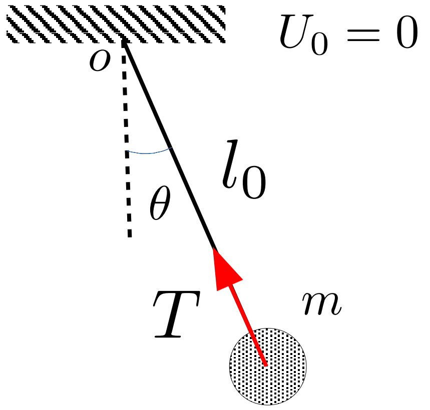
$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) - mgx\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} + \lambda\frac{\partial f}{\partial\theta}$$

$$\ddot{x} = -g\sin\theta_0 \qquad \lambda = mgx\cos\theta_0$$

$$N = \frac{\lambda}{x} = mg\cos\theta_0$$

مکانیک لاگرانژی



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \quad V = -mgr\cos\theta$$

$$f(r) = r - l_0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

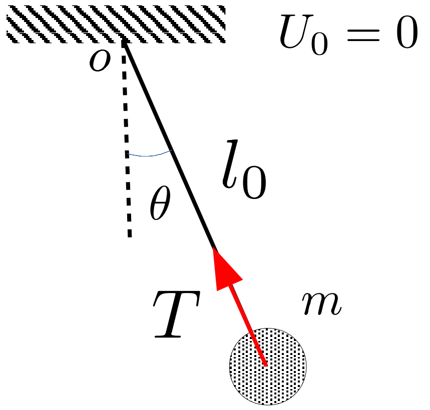
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta}$$

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta + \lambda \qquad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta$$

$$f(r) = r - l_0 = 0 \Rightarrow r = l_0 \Rightarrow \ddot{r} = \dot{r} = 0$$

$$0 = ml_0\dot{\theta}^2 + mg\cos\theta + \lambda \qquad \ddot{\theta} = -\frac{g}{l_0}\sin\theta$$

مکانیک لاگرانژی



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \quad V = -mgr\cos\theta$$

$$f(r) = r - l_0$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

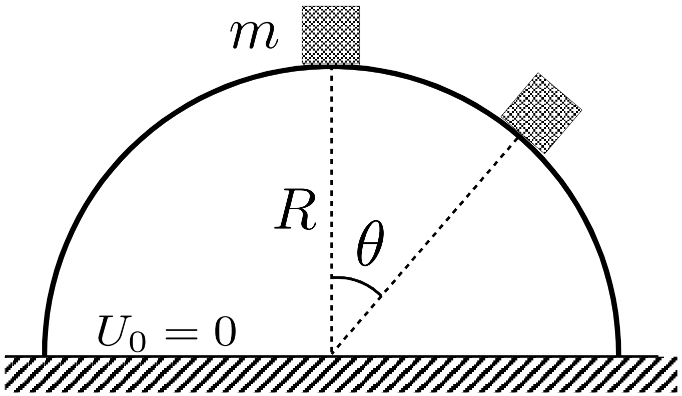
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta}$$

$$0 = ml_0\dot{\theta}^2 + mg\cos\theta + \lambda \qquad \ddot{\theta} = -\frac{g}{l_0}\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l_0}\sin\theta \Rightarrow \dot{\theta}\mathrm{d}\dot{\theta} = -\frac{g}{l_0}\sin\theta\mathrm{d}\theta \Rightarrow \frac{1}{2}\dot{\theta}^2 = \frac{g}{l_0}(\cos\theta - cos\alpha)$$

$$\lambda = -mg(3\cos\theta - 2\cos\alpha) \Rightarrow T = -\lambda = mg(3\cos\theta - 2\cos\alpha)$$

مکانیک لاگرانژی



$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2),\qquad V=mgr\cos\theta$$

$$f=r-R$$

$$\mathcal{L}=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-mgr\cos\theta$$

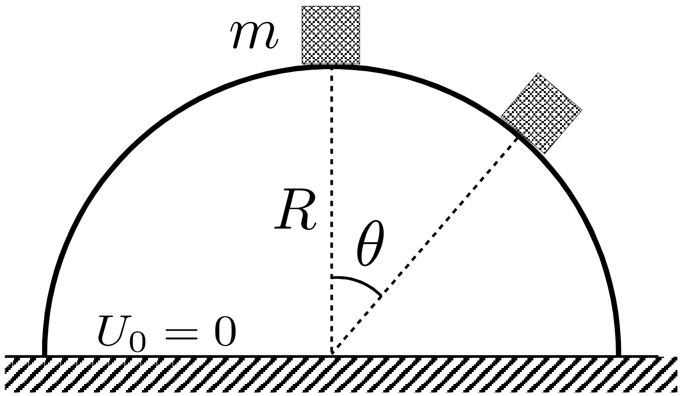
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right)=\frac{\partial\mathcal{L}}{\partial r}+\lambda\frac{\partial f}{\partial r}\qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)=\frac{\partial\mathcal{L}}{\partial\theta}+\lambda\frac{\partial f}{\partial\theta}$$

$$m\ddot{r}=mr\dot{\theta}^2-mg\cos\theta+\lambda\qquad\qquad mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}=mgr\sin\theta$$

$$f(r)=r-R=0\Rightarrow r=R\Rightarrow\ddot{r}=\dot{r}=0$$

$$0=mR\dot{\theta}^2-mg\cos\theta+\lambda\qquad\qquad \ddot{\theta}=\frac{g}{R}\sin\theta$$

مکانیک لاگرانژی



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \qquad V = mgr\cos\theta$$

$$f = r - R$$

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

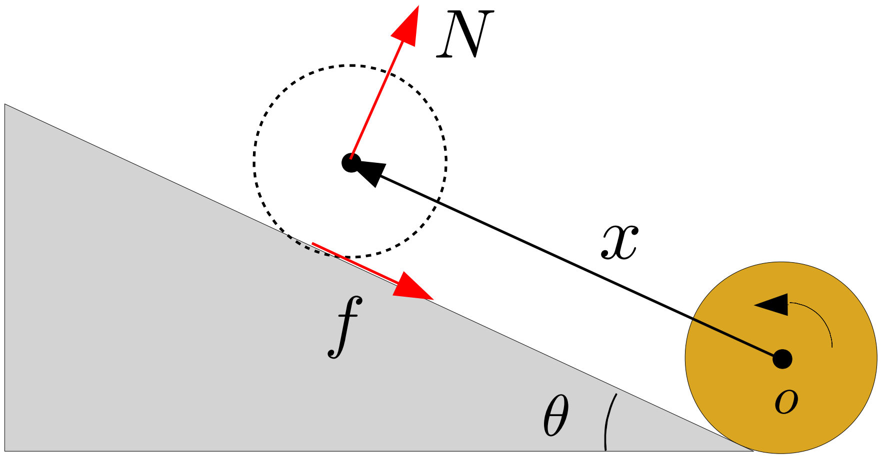
$$0 = mR\dot{\theta}^2 - mg\cos\theta + \lambda \qquad\qquad \ddot{\theta} = \frac{g}{R}\sin\theta$$

$$\ddot{\theta} = \frac{g}{R}\sin\theta \Rightarrow \dot{\theta}\mathrm{d}\dot{\theta} = \frac{g}{R}\sin\theta\mathrm{d}\theta \Rightarrow \frac{1}{2}\dot{\theta}^2 = \frac{g}{R}(1-\cos\theta)$$

$$\lambda = 3mg\cos\theta - 2mg = N$$

$$N = 0 \Rightarrow \cos\theta = \frac{2}{3}$$

مکانیک لاگرانژی



$$T=\frac{1}{2}M(\dot{x}^2+x^2\dot{\theta}^2)+\frac{1}{2}\mathbb{I}_o\dot{\phi}^2,$$

$$V=Mgx\sin\theta$$

$$f_1=x-R\phi \qquad f_2=\theta-\theta_0$$

$$\mathcal{L}=\frac{1}{2}M(\dot{x}^2+x^2\dot{\theta}^2)+\frac{1}{4}MR^2\dot{\phi}^2-Mgx\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)=\frac{\partial\mathcal{L}}{\partial x}+\lambda_1\frac{\partial f_1}{\partial x}\Rightarrow M\ddot{x}=Mx\dot{\theta}^2-Mg\sin\theta+\lambda_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)=\frac{\partial\mathcal{L}}{\partial\phi}+\lambda_1\frac{\partial f_1}{\partial\phi}\Rightarrow\frac{1}{2}MR^2\ddot{\phi}=-R\lambda_1\Rightarrow\frac{1}{2}MR\ddot{\phi}=-\lambda_1$$

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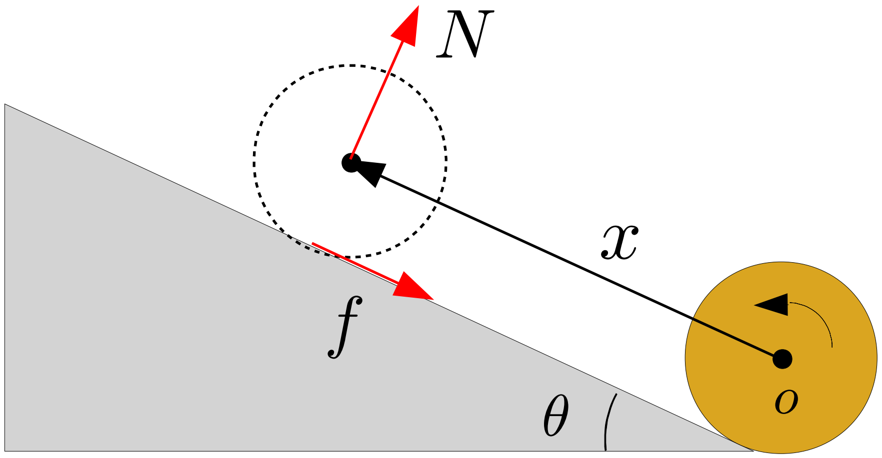

$$T = \frac{1}{2}M(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{2}\mathbb{I}_o\dot{\phi}^2,$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} + \lambda_2\frac{\partial f_2}{\partial\theta} \Rightarrow Mx^2\ddot{\theta} + 2Mx\dot{x}\dot{\theta} = -Mgx\cos\theta + \lambda_2$$

$$f_1 = x - R\phi = 0 \Rightarrow \dot{x} = R\dot{\phi} \Rightarrow \ddot{x} = R\ddot{\phi}$$

$$f_2 = \theta - \theta_0 = 0 \Rightarrow \dot{\theta} = \ddot{\theta} = 0$$

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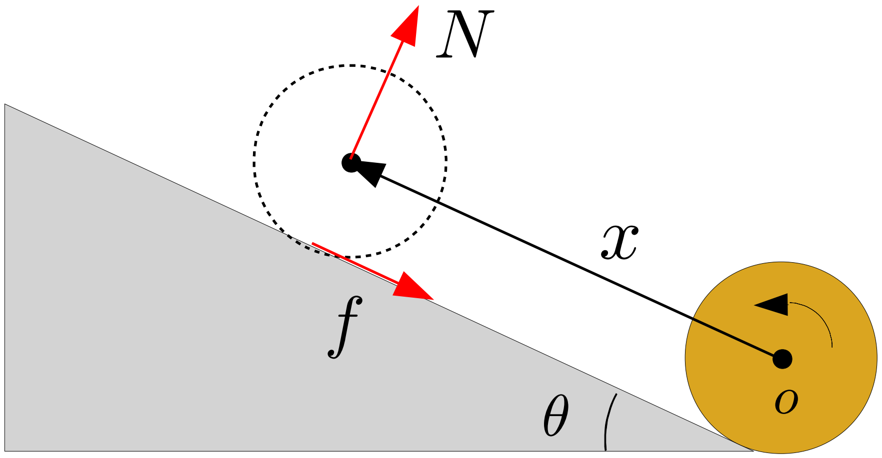

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$$\begin{cases} M\ddot{x} = Mx\cancelto{0}{\dot{\theta}^2} - Mg\sin\theta_0 + \lambda_1 \\ \frac{1}{2}MR\ddot{\phi} = -\lambda_1 \\ Mx^2\cancelto{0}{\ddot{\theta}} + 2Mx\dot{x}\cancelto{0}{\dot{\theta}} = -Mgx\cos\theta_0 + \lambda_2 \end{cases}$$

$$\begin{cases} M\ddot{x} = -Mg\sin\theta_0 + \lambda_1 \\ \frac{1}{2}M\ddot{x} = -\lambda_1 \\ 0 = -Mgx\cos\theta_0 + \lambda_2 \end{cases}$$

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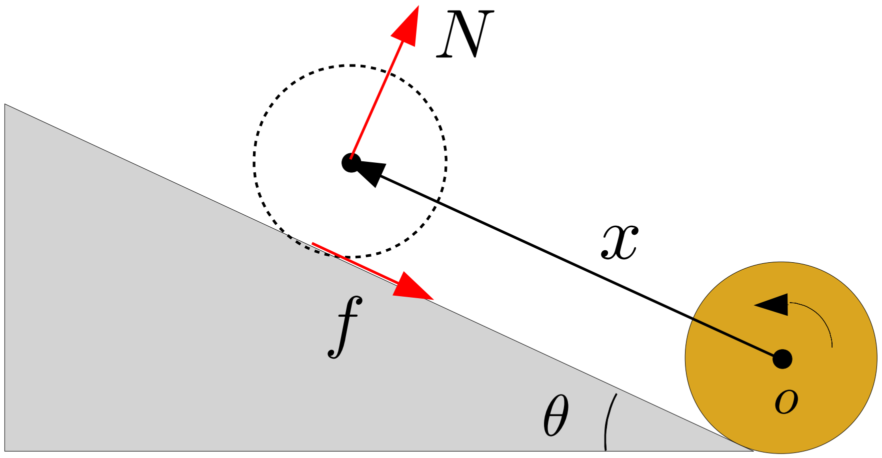

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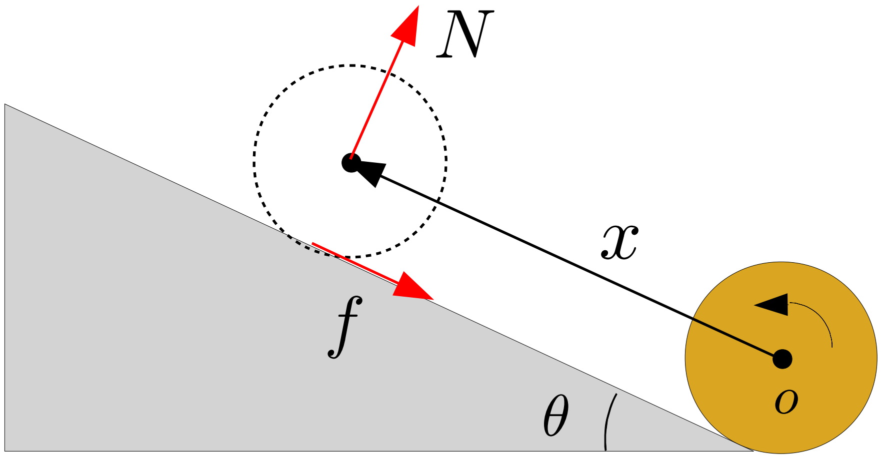

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} + \lambda_1\frac{\partial f_1}{\partial\dot{x}} \Rightarrow M\ddot{x} = Mx\dot{\theta}^2 - Mg\sin\theta + \lambda_1$$

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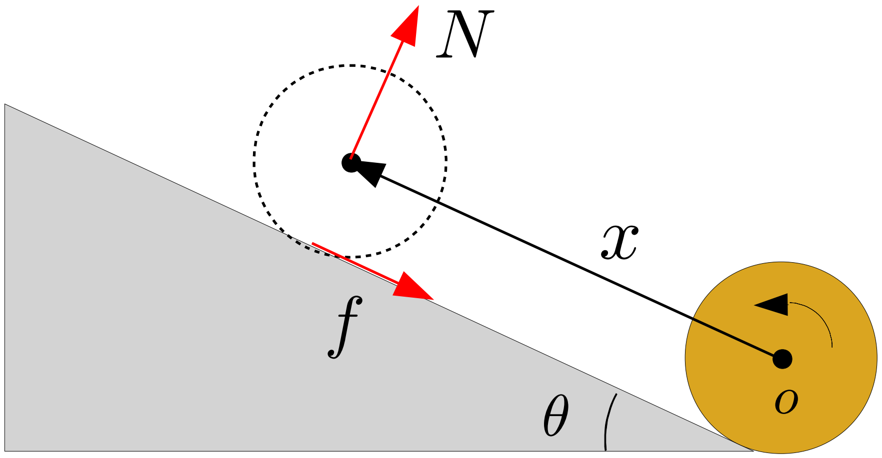
مکانیک لاگرانژی



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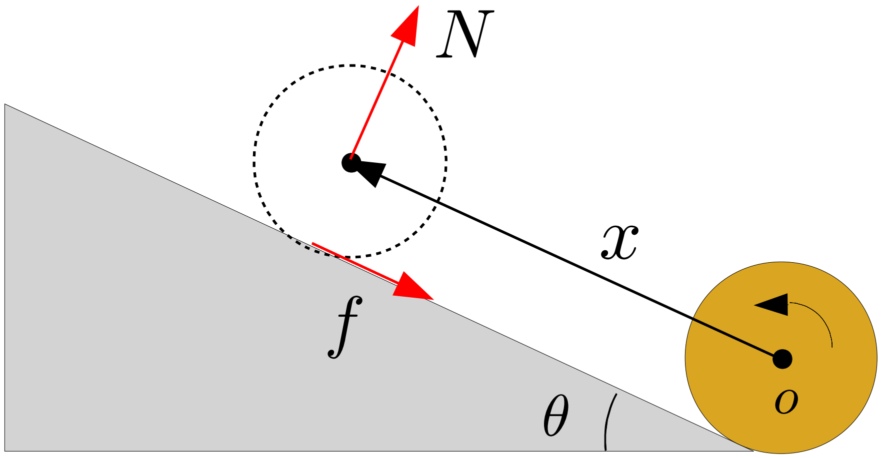
مکانیک لاگرانژی



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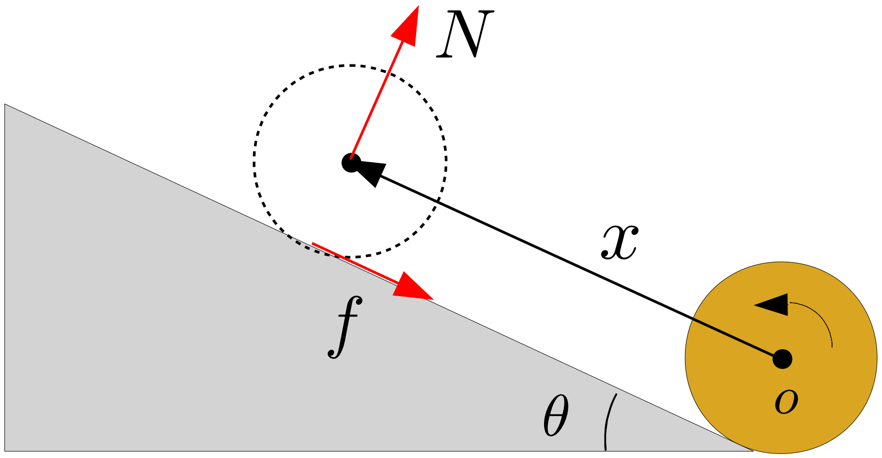

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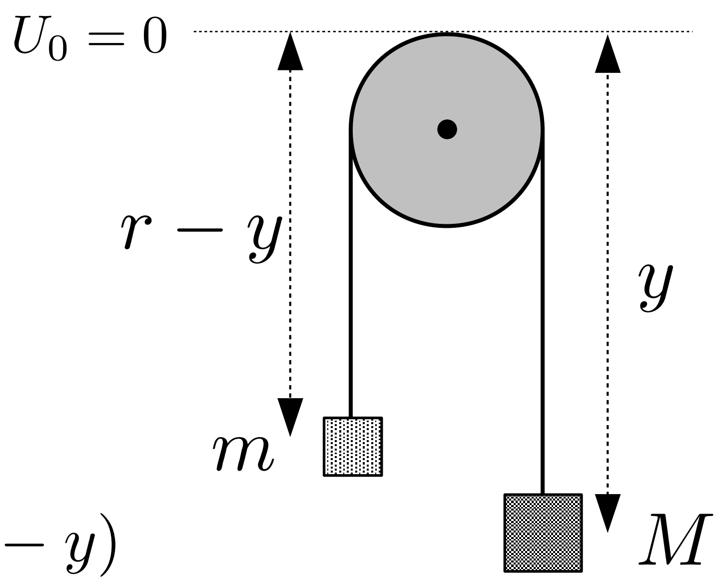
مکانیک لاگرانژی

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{r} - \dot{y})^2$$

$$V = -Mgy - mg(r - y)$$



$$f = r - l$$

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{r} - \dot{y})^2 + Mgy + mg(r - y)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\right] = \frac{\partial \mathcal{L}}{\partial y} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\partial \mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r}$$

$$(M + m)\ddot{y} - m\ddot{r} = (M - m)g \qquad m\ddot{r} - m\ddot{y} = mg + \lambda$$

$$f = r - l = 0 \Rightarrow r = l \Rightarrow \ddot{r} = \dot{r} = 0$$

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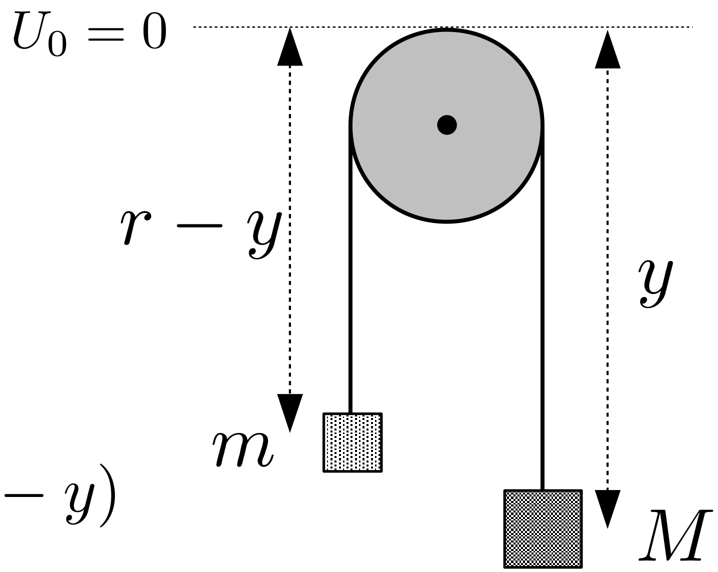

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{r}}\right] = \frac{\partial \mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r}$$

$$(M + m)\ddot{y} = (M - m)g$$

$$\lambda = -m(\ddot{y} + g)$$

$$\ddot{x} = \frac{M - m}{M + m}g \qquad \lambda = -\frac{Mm}{M + m}g$$