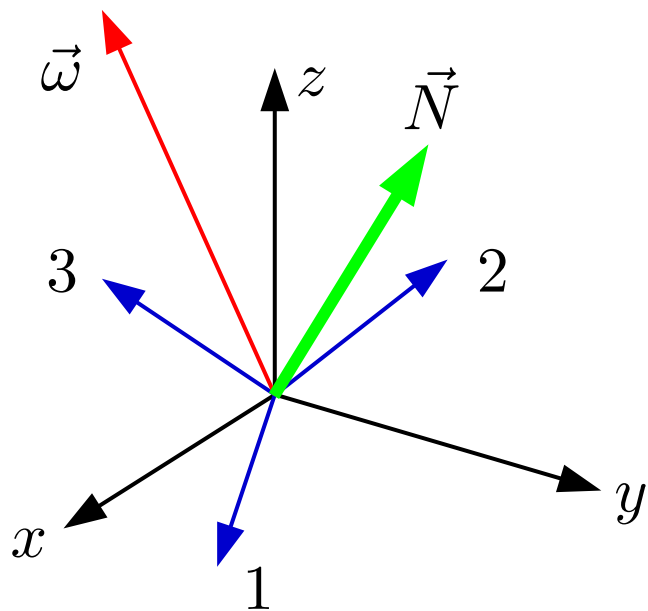


جلسه بیست و پنجم

مکانیک تحلیلی

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گروه فیزیک، دانشکده علوم پایه
دانشگاه قم
اسفند ۹۸

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات لخت (ثابت) xyz

$$\vec{N} = \frac{d\vec{L}}{dt}$$

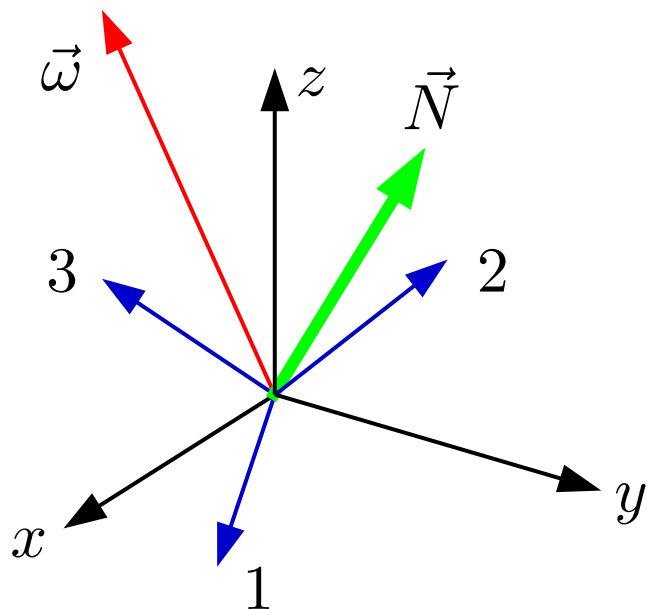
یادآوری: رابطه مشتق زمانی در دستگاه مختصات لخت (ثابت) و دستگاه مختصات نا لخت (چرخان)

$$\left(\frac{d}{dt}\right)_{\text{ثابت}} = \left(\frac{d}{dt}\right)_{\text{چرخان}} + \vec{\omega} \times$$

* دینامیک جسم صلب در دستگاه مختصات نا لخت (چرخان) 123

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{ثابت}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{چرخان}} + \vec{\omega} \times \vec{L}$$

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات لخت (ثابت) xyz

$$\vec{N} = \frac{d\vec{L}}{dt}$$

* دینامیک جسم صلب در دستگاه مختصات نا لخت (چرخان) 123

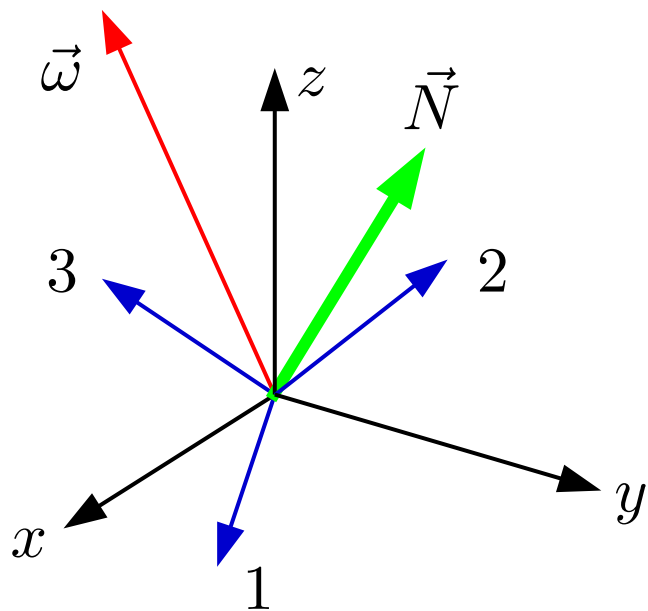
$$\left(\frac{d\vec{L}}{dt}\right)_{\text{ثابت}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{چرخان}} + \vec{\omega} \times \vec{L}$$

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{\text{چرخان}} + \vec{\omega} \times \vec{L} \Rightarrow \left(\frac{d\vec{L}}{dt}\right)_{\text{چرخان}} = \vec{N} - \vec{\omega} \times \vec{L}$$

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3, \quad \vec{N} = N_1 \hat{e}_1 + N_2 \hat{e}_2 + N_3 \hat{e}_3$$

$$\vec{L} = \mathbb{I}_1 \omega_1 \hat{e}_1 + \mathbb{I}_2 \omega_2 \hat{e}_2 + \mathbb{I}_3 \omega_3 \hat{e}_3$$

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات ناخست (چرخان) 123

$$\left(\frac{d\vec{L}}{dt} \right)_{\text{چرخان}} = \vec{N} - \vec{\omega} \times \vec{L}$$

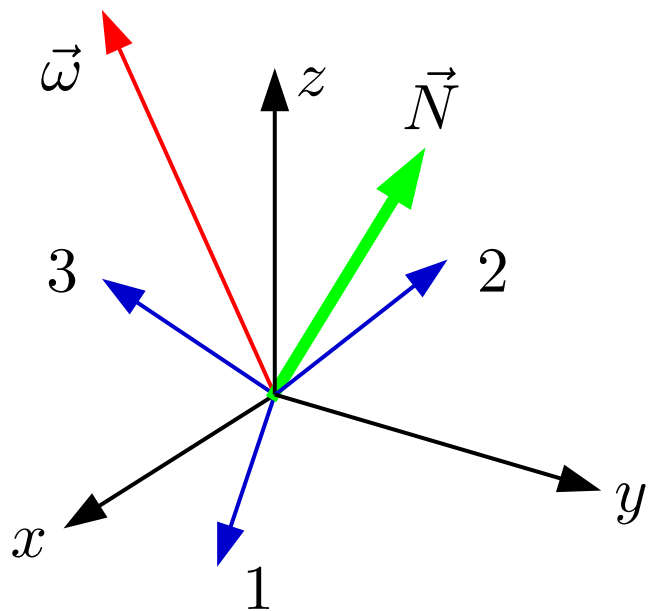
$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3, \quad \vec{N} = N_1 \hat{e}_1 + N_2 \hat{e}_2 + N_3 \hat{e}_3$$

$$\vec{L} = \mathbb{I}_1 \omega_1 \hat{e}_1 + \mathbb{I}_2 \omega_2 \hat{e}_2 + \mathbb{I}_3 \omega_3 \hat{e}_3$$

$$\left(\frac{d\vec{L}}{dt} \right)_{\text{چرخان}} = \mathbb{I}_1 \dot{\omega}_1 \hat{e}_1 + \mathbb{I}_2 \dot{\omega}_2 \hat{e}_2 + \mathbb{I}_3 \dot{\omega}_3 \hat{e}_3$$

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \mathbb{I}_1 \omega_1 & \mathbb{I}_2 \omega_2 & \mathbb{I}_3 \omega_3 \end{vmatrix} = (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \hat{e}_1 + (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \hat{e}_2 + (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \hat{e}_3$$

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات ناخست (چرخان) 123

$$\left(\frac{d\vec{L}}{dt} \right)_{\text{چرخان}} = \vec{N} - \vec{\omega} \times \vec{L}$$

$$\left(\frac{d\vec{L}}{dt} \right)_{\text{چرخان}} = \mathbb{I}_1 \dot{\omega}_1 \hat{e}_1 + \mathbb{I}_2 \dot{\omega}_2 \hat{e}_2 + \mathbb{I}_3 \dot{\omega}_3 \hat{e}_3$$

$$\vec{N} = N_1 \hat{e}_1 + N_2 \hat{e}_2 + N_3 \hat{e}_3$$

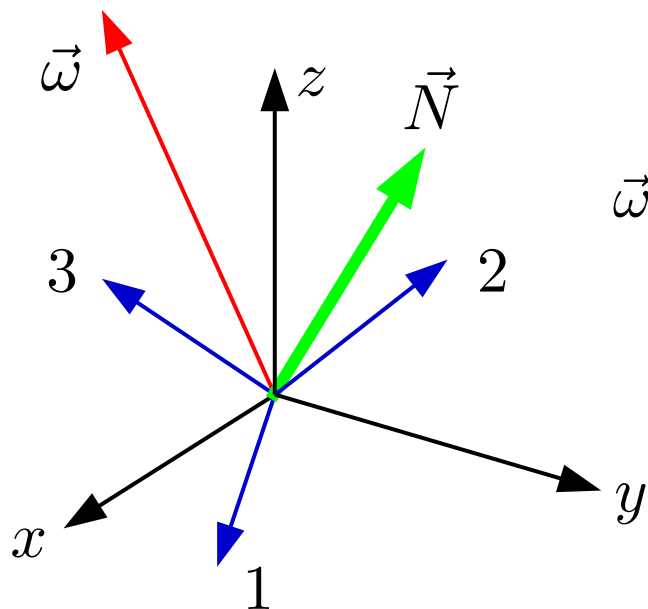
$$\vec{\omega} \times \vec{L} = (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \hat{e}_1 + (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \hat{e}_2 + (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \hat{e}_3$$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = N_1 - (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = N_2 - (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = N_3 - (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

هدف پیدا کردن:

$$\omega_1, \omega_2, \omega_3$$

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات نالخت (چرخان) 123

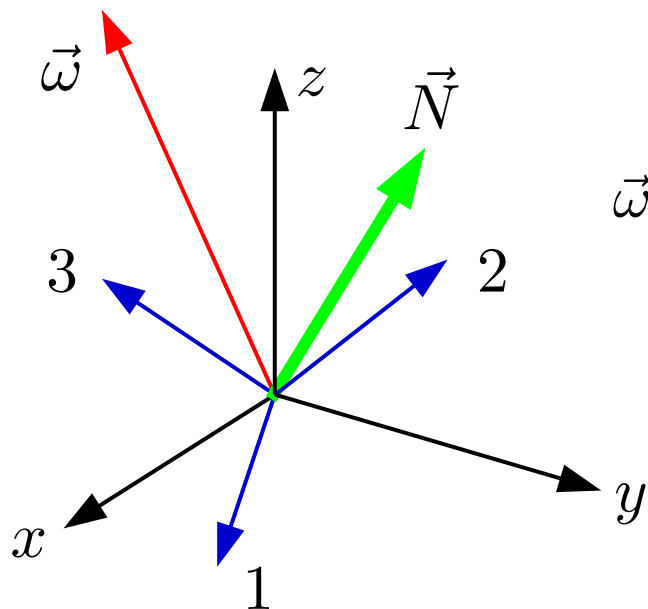
$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3, \quad \vec{N} = N_1 \hat{e}_1 + N_2 \hat{e}_2 + N_3 \hat{e}_3$$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = N_1 - (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = N_2 - (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = N_3 - (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

□ حالت خاص ۱: $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

$$\begin{cases} 0 = N_1 - (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ 0 = N_2 - (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ 0 = N_3 - (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases} \Rightarrow \begin{cases} N_1 = (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ N_2 = (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ N_3 = (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

حرکت اجسام صلب در سه بعد



* دینامیک جسم صلب در دستگاه مختصات ناخست (چرخان) 123

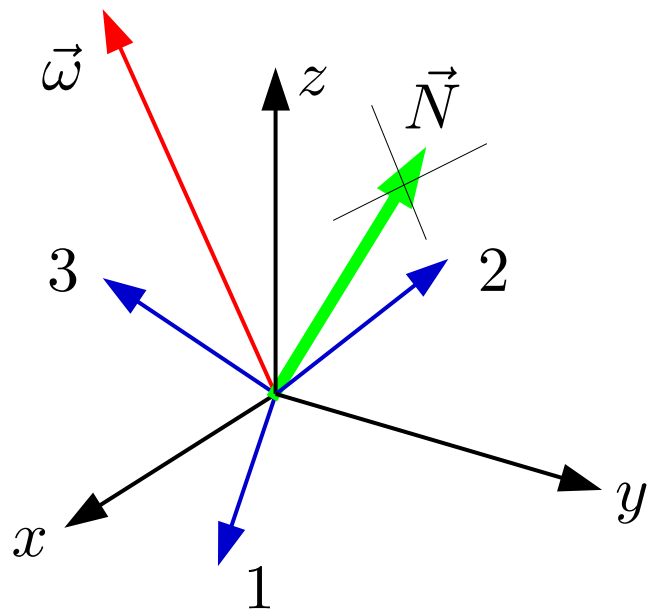
$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3, \quad \vec{N} = N_1 \hat{e}_1 + N_2 \hat{e}_2 + N_3 \hat{e}_3$$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = N_1 - (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = N_2 - (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = N_3 - (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

□ حالت خاص ۲: $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$ و $\mathbb{I}_1 = \mathbb{I}_2 = \mathbb{I}_3$

$$\begin{cases} N_1 = (\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ N_2 = (\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ N_3 = (\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases} \implies \begin{cases} N_1 = 0 \\ N_2 = 0 \\ N_3 = 0 \end{cases} \implies \text{تعادل دینامیکی}$$

حرکت اجسام صلب در سه بعد



چرخش آزاد

$$\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

هدف پیدا کردن:

$$\omega_1, \omega_2, \omega_3$$

ثوابت حرکت

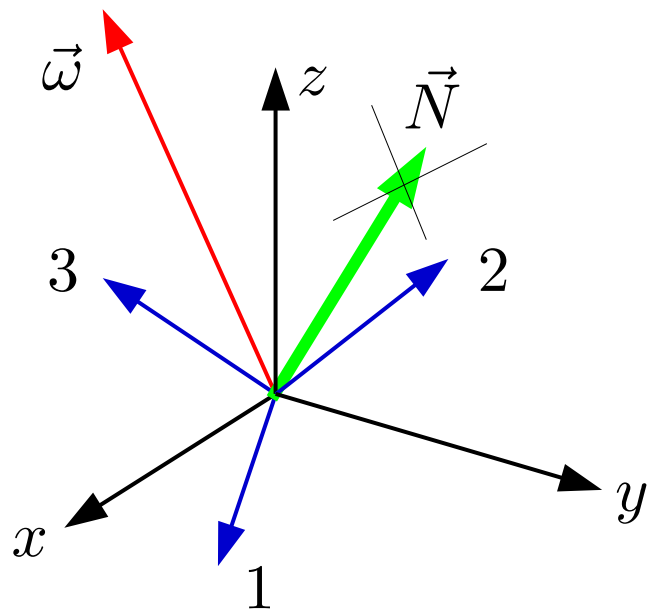
* انرژی جنبشی

$$\frac{1}{2} \mathbb{I}_1 \omega_1^2 + \frac{1}{2} \mathbb{I}_2 \omega_2^2 + \frac{1}{2} \mathbb{I}_3 \omega_3^2 = T = \text{const.}$$

* مربع اندازه حرکت زاویه‌ای

$$\mathbb{I}_1^2 \omega_1^2 + \mathbb{I}_2^2 \omega_2^2 + \mathbb{I}_3^2 \omega_3^2 = L^2 = \text{const.}$$

حرکت اجسام صلب در سه بعد



چرخش آزاد $\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

ثوابت حرکت

$$T = \frac{1}{2} \mathbb{I}_1 \omega_1^2 + \frac{1}{2} \mathbb{I}_2 \omega_2^2 + \frac{1}{2} \mathbb{I}_3 \omega_3^2$$

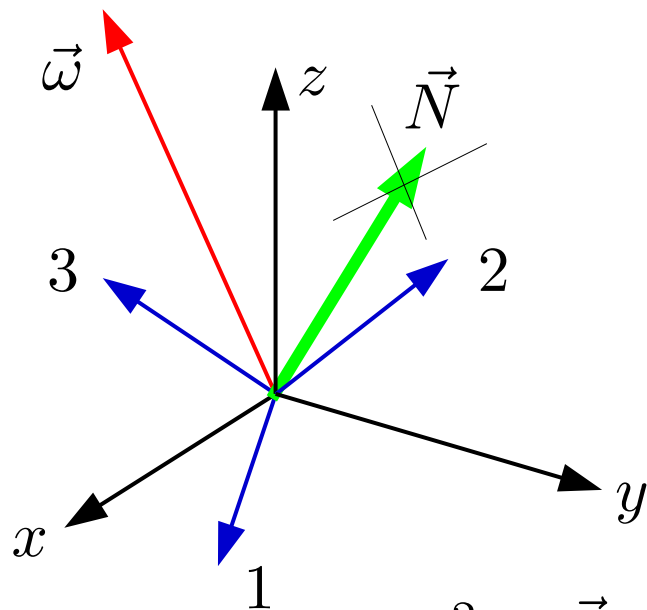
* انرژی جنبشی

$$\frac{dT}{dt} = \mathbb{I}_1 \omega_1 \dot{\omega}_1 + \mathbb{I}_2 \omega_2 \dot{\omega}_2 + \mathbb{I}_3 \omega_3 \dot{\omega}_3 = (\mathbb{I}_1 \dot{\omega}_1) \omega_1 + (\mathbb{I}_2 \dot{\omega}_2) \omega_2 + (\mathbb{I}_3 \dot{\omega}_3) \omega_3$$

$$= -(\cancel{\mathbb{I}_3} - \cancel{\mathbb{I}_2}) \omega_2 \omega_3 \omega_1 - (\cancel{\mathbb{I}_1} - \cancel{\mathbb{I}_3}) \omega_3 \omega_1 \omega_2 - (\cancel{\mathbb{I}_2} - \cancel{\mathbb{I}_1}) \omega_1 \omega_2 \omega_3 = 0$$

$$\frac{dT}{dt} = 0 \Rightarrow \text{انرژی جنبشی ثابت حرکت است}$$

حرکت اجسام صلب در سه بعد



چرخش آزاد $\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

ثوابت حرکت

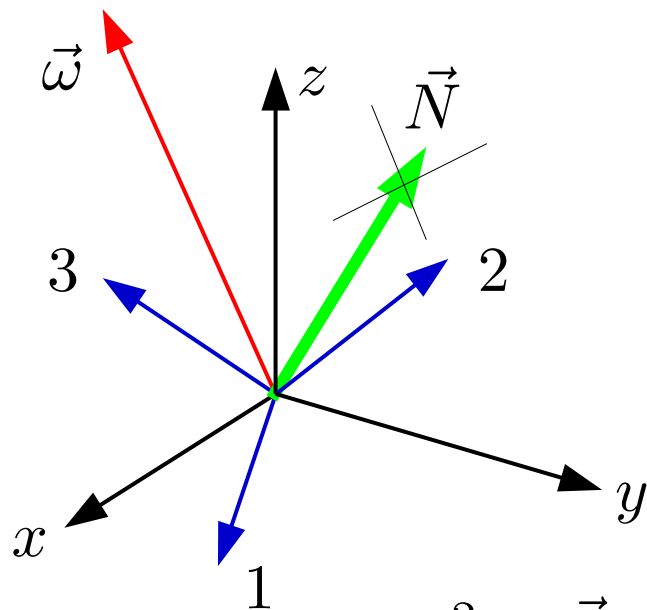
* مربع اندازه حرکت زاویه‌ای $L^2 = \vec{L} \cdot \vec{L} = \mathbb{I}_1^2 \omega_1^2 + \mathbb{I}_2^2 \omega_2^2 + \mathbb{I}_3^2 \omega_3^2$

$$\frac{dL^2}{dt} = 2\mathbb{I}_1^2 \omega_1 \dot{\omega}_1 + 2\mathbb{I}_2^2 \omega_2 \dot{\omega}_2 + 2\mathbb{I}_3^2 \omega_3 \dot{\omega}_3$$

$$= 2(\mathbb{I}_1 \dot{\omega}_1)(\mathbb{I}_1 \omega_1) + 2(\mathbb{I}_2 \dot{\omega}_2)(\mathbb{I}_2 \omega_2) + 2(\mathbb{I}_3 \dot{\omega}_3)(\mathbb{I}_3 \omega_3)$$

$$= -2(\cancel{\mathbb{I}_1 \mathbb{I}_3} - \cancel{\mathbb{I}_1 \mathbb{I}_2}) \omega_2 \omega_3 \omega_1 - 2(\cancel{\mathbb{I}_2 \mathbb{I}_1} - \cancel{\mathbb{I}_2 \mathbb{I}_3}) \omega_3 \omega_1 \omega_2 - 2(\cancel{\mathbb{I}_3 \mathbb{I}_2} - \cancel{\mathbb{I}_3 \mathbb{I}_1}) \omega_1 \omega_2 \omega_3 = 0$$

حرکت اجسام صلب در سه بعد



چرخش آزاد $\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

ثوابت حرکت

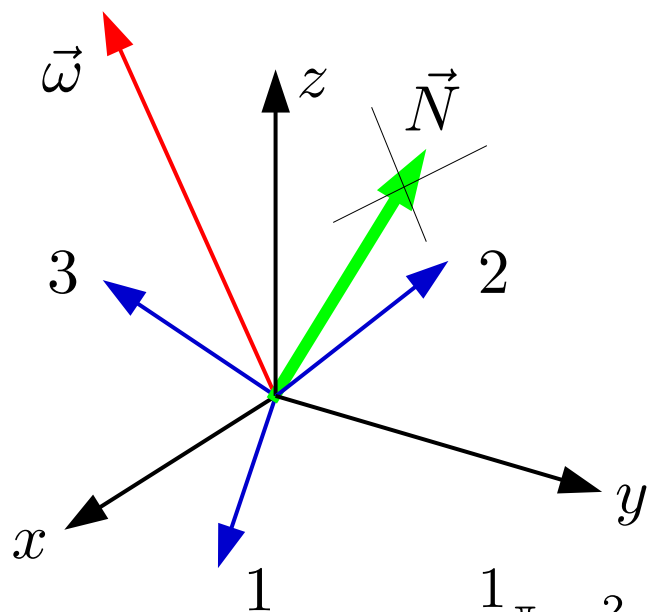
* مربع اندازه حرکت زاویه‌ای $L^2 = \vec{L} \cdot \vec{L} = \mathbb{I}_1^2 \omega_1^2 + \mathbb{I}_2^2 \omega_2^2 + \mathbb{I}_3^2 \omega_3^2$

$$\frac{dL^2}{dt} = 2\mathbb{I}_1^2 \omega_1 \dot{\omega}_1 + 2\mathbb{I}_2^2 \omega_2 \dot{\omega}_2 + 2\mathbb{I}_3^2 \omega_3 \dot{\omega}_3$$

$$= -2(\cancel{\mathbb{I}_1 \mathbb{I}_3} - \cancel{\mathbb{I}_1 \mathbb{I}_2}) \omega_2 \omega_3 \omega_1 - 2(\cancel{\mathbb{I}_2 \mathbb{I}_1} - \cancel{\mathbb{I}_2 \mathbb{I}_3}) \omega_3 \omega_1 \omega_2 - 2(\cancel{\mathbb{I}_3 \mathbb{I}_2} - \cancel{\mathbb{I}_3 \mathbb{I}_1}) \omega_1 \omega_2 \omega_3 = 0$$

$$\frac{dL^2}{dt} = 0 \Rightarrow \text{مربع اندازه حرکت زاویه‌ای ثابت حرکت است}$$

حرکت اجسام صلب در سه بعد



$$\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$$

چرخش آزاد

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases}$$

هدف پیدا کردن:

$$\omega_1, \omega_2, \omega_3$$

ثوابت حرکت

$$\frac{1}{2} \mathbb{I}_1 \omega_1^2 + \frac{1}{2} \mathbb{I}_2 \omega_2^2 + \frac{1}{2} \mathbb{I}_3 \omega_3^2 = T = \text{const.}$$

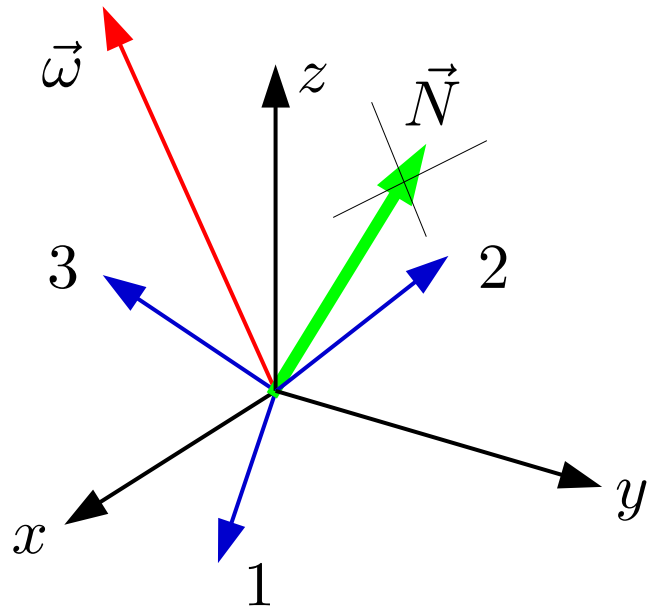
* انرژی جنبشی

$$\frac{\omega_1^2}{2T/\mathbb{I}_1} + \frac{\omega_2^2}{2T/\mathbb{I}_2} + \frac{\omega_3^2}{2T/\mathbb{I}_3} = 1$$

* مربع اندازه حرکت زاویه‌ای

$$\mathbb{I}_1^2 \omega_1^2 + \mathbb{I}_2^2 \omega_2^2 + \mathbb{I}_3^2 \omega_3^2 = L^2 = \text{const.} \Rightarrow \frac{\omega_1^2}{L^2/\mathbb{I}_1^2} + \frac{\omega_2^2}{L^2/\mathbb{I}_2^2} + \frac{\omega_3^2}{L^2/\mathbb{I}_3^2} = 1$$

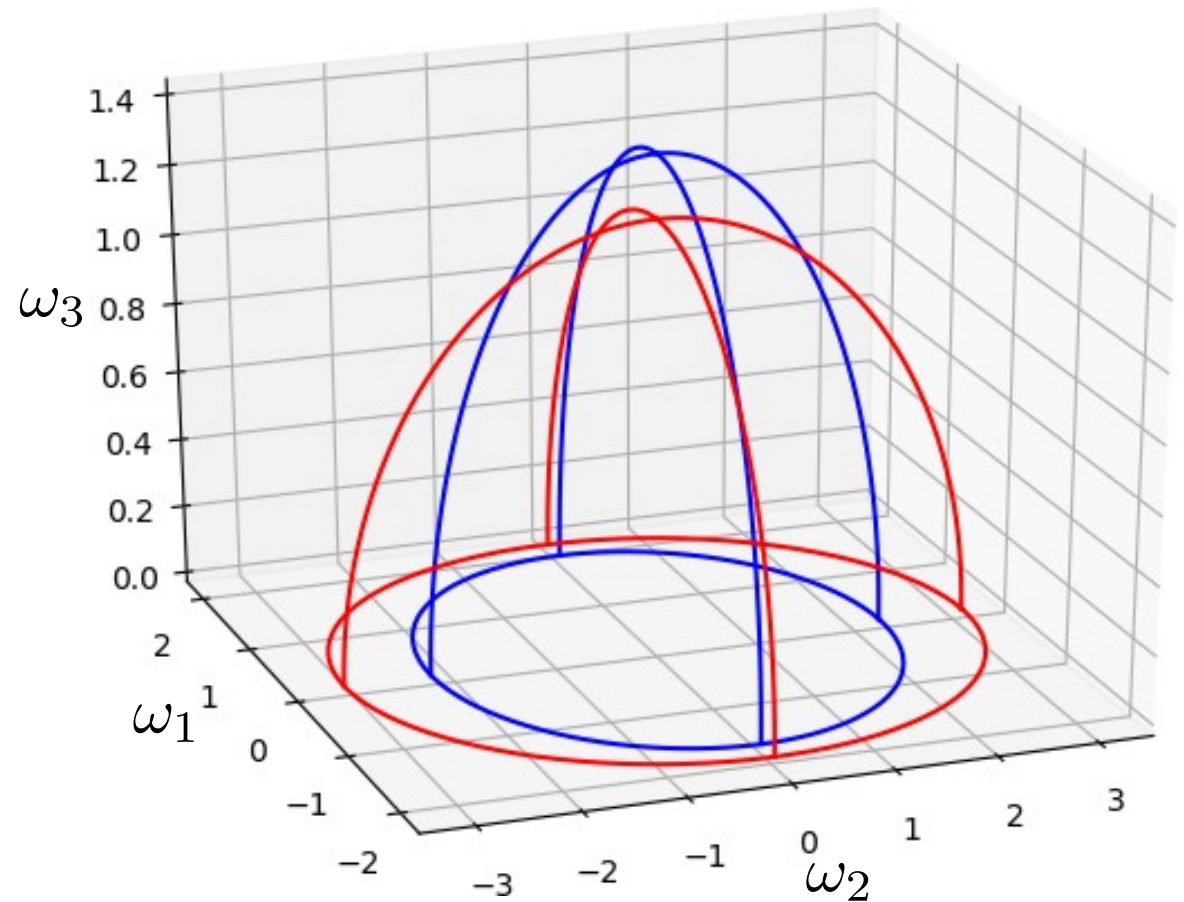
حرکت اجسام صلب در سه بعد



چرخش آزاد $\vec{N} = 0 \Rightarrow N_1 = N_2 = N_3 = 0$

$$\frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = 1$$

$$\frac{\omega_1^2}{L^2/I_1^2} + \frac{\omega_2^2}{L^2/I_2^2} + \frac{\omega_3^2}{L^2/I_3^2} = 1$$



حرکت اجسام صلب در سه بعد

$$\frac{\omega_1^2}{2T/I_1} + \frac{\omega_2^2}{2T/I_2} + \frac{\omega_3^2}{2T/I_3} = 1, \quad \frac{\omega_1^2}{L^2/I_1^2} + \frac{\omega_2^2}{L^2/I_2^2} + \frac{\omega_3^2}{L^2/I_3^2} = 1 \quad \text{چرخش آزاد}$$

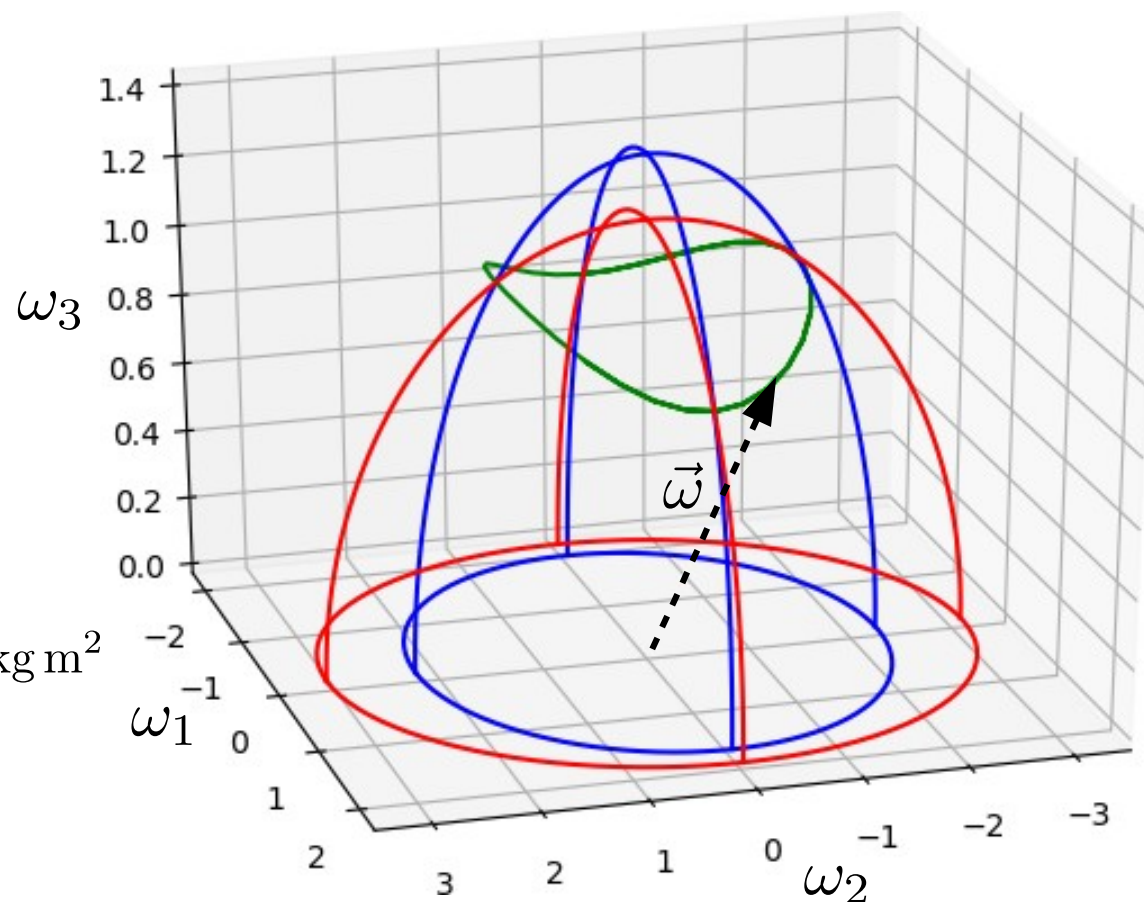
$$\begin{cases} I_1 \dot{\omega}_1 = -(I_3 - I_2) \omega_2 \omega_3 \\ I_2 \dot{\omega}_2 = -(I_1 - I_3) \omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = -(I_2 - I_1) \omega_1 \omega_2 \end{cases}$$

$$\begin{cases} \dot{\omega}_1 = [(I_2 - I_3)/I_1] \omega_2 \omega_3 \\ \dot{\omega}_2 = [(I_3 - I_1)/I_2] \omega_3 \omega_1 \\ \dot{\omega}_3 = [(I_1 - I_2)/I_3] \omega_1 \omega_2 \end{cases}$$

$$I_1 = 0.2 \text{ kg m}^2, \quad I_2 = 0.3 \text{ kg m}^2, \quad I_3 = 0.5 \text{ kg m}^2$$

$$[\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}] = [1, 1, 1] \text{ s}^{-1}$$

$$T = 0.5 \text{ J}, \quad L^2 = 0.38 \text{ J s}$$

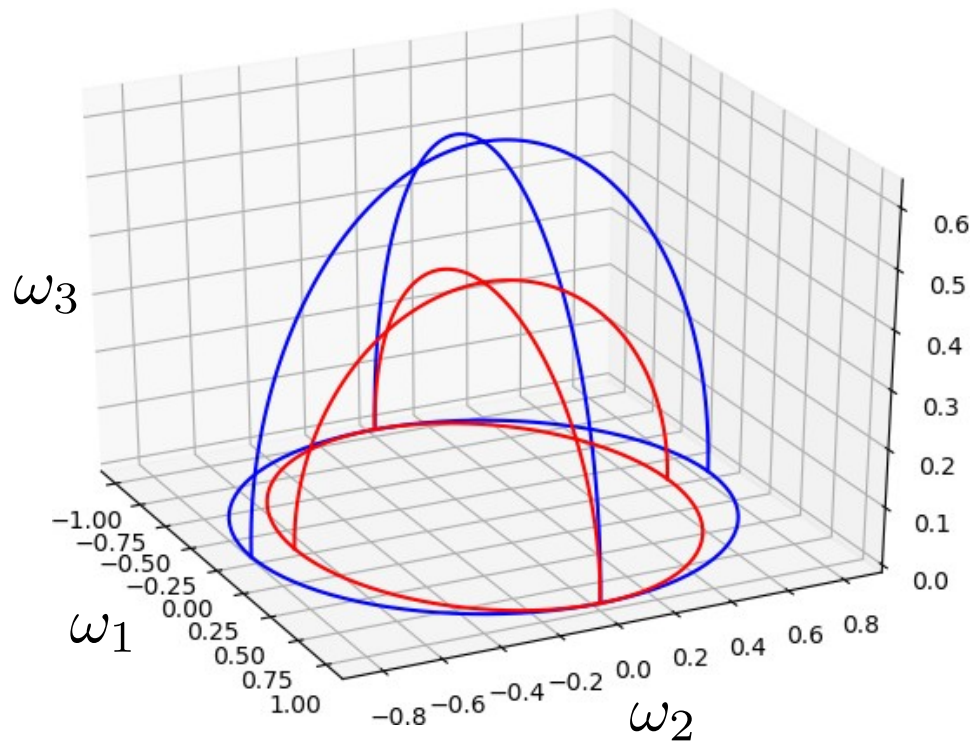


حرکت اجسام صلب در سه بعد

چرخش آزاد $\mathbb{I}_1 \leq \mathbb{I}_2 \leq \mathbb{I}_3$

$$[\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}] = [0, 0, 1]s^{-1}$$

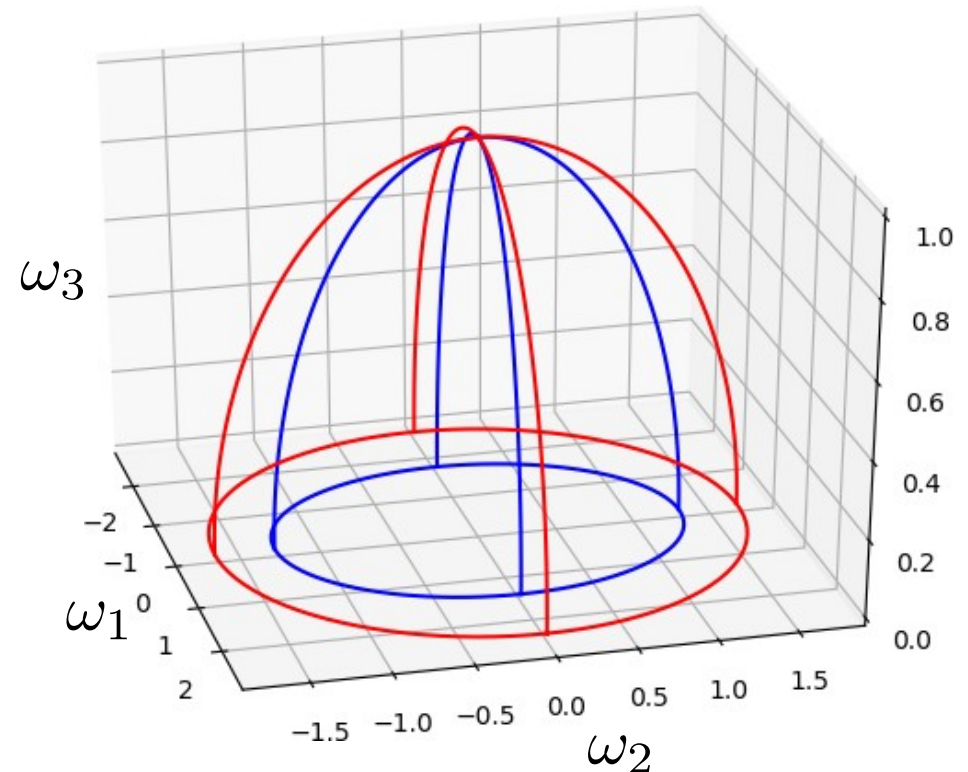
در جهت محور 3



$$[\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}] = [1, 0, 0]s^{-1}$$

در جهت محور 1

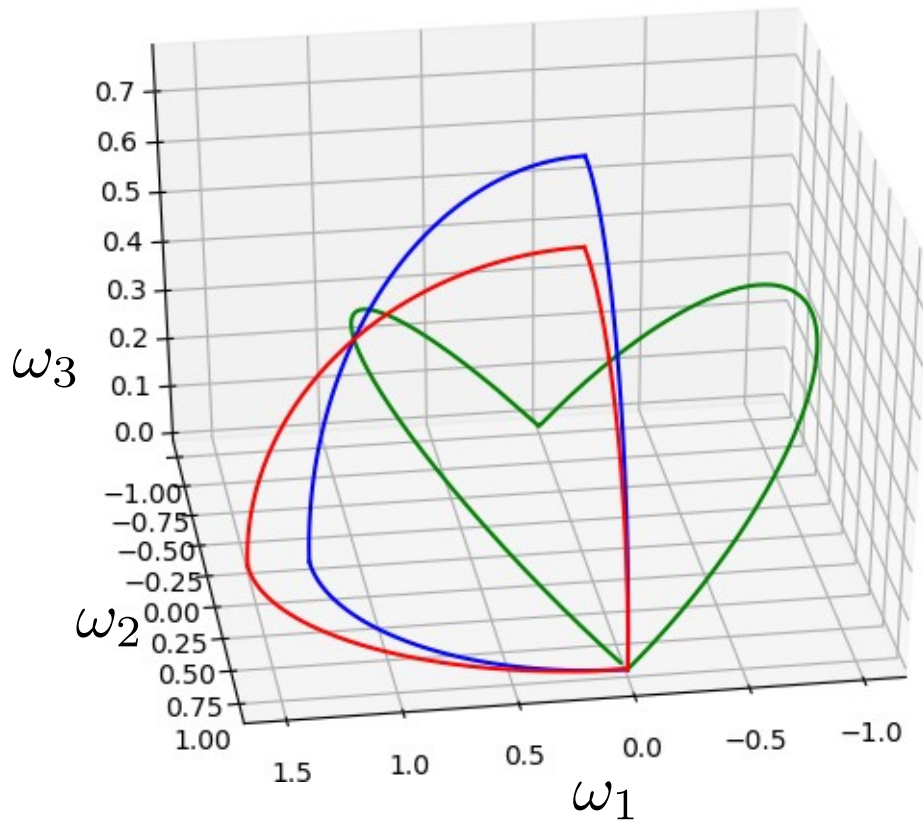
پایدار



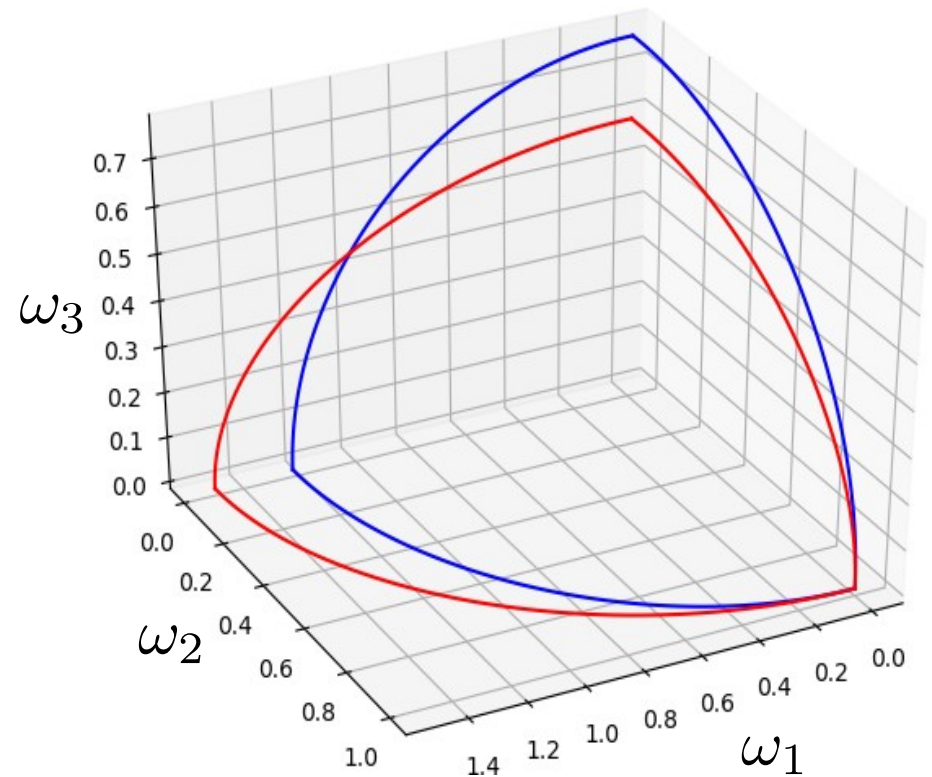
حرکت اجسام صلب در سه بعد

چرخش آزاد $\mathbb{I}_1 \leq \mathbb{I}_2 \leq \mathbb{I}_3$

در جهت محور 2



$$[\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}] = [0, 1, 0] \text{s}^{-1} \quad \text{ناپایدار}$$



حرکت اجسام صلب در سه بعد

چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases} \Rightarrow \begin{cases} \mathbb{I} \dot{\omega}_1 = -(\mathbb{I}_s - \mathbb{I}) \omega_2 \omega_3 & \text{(a)} \\ \mathbb{I} \dot{\omega}_2 = -(\mathbb{I} - \mathbb{I}_s) \omega_3 \omega_1 & \text{(b)} \\ \mathbb{I}_s \dot{\omega}_3 = 0 \end{cases}$$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ثابت}$$

$$\text{(a)} \quad \mathbb{I} \dot{\omega}_1 + (\mathbb{I}_s - \mathbb{I}) \omega_3 \omega_2 = 0 \Rightarrow \dot{\omega}_1 + \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3 \omega_2 = 0$$

$$\text{(b)} \quad \mathbb{I} \dot{\omega}_2 + (\mathbb{I} - \mathbb{I}_s) \omega_3 \omega_1 = 0 \Rightarrow \dot{\omega}_2 - \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3 \omega_1 = 0$$

$$\Omega = \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3$$

حرکت اجسام صلب در سه بعد

چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

$$\begin{cases} \mathbb{I}_1 \dot{\omega}_1 = -(\mathbb{I}_3 - \mathbb{I}_2) \omega_2 \omega_3 \\ \mathbb{I}_2 \dot{\omega}_2 = -(\mathbb{I}_1 - \mathbb{I}_3) \omega_3 \omega_1 \\ \mathbb{I}_3 \dot{\omega}_3 = -(\mathbb{I}_2 - \mathbb{I}_1) \omega_1 \omega_2 \end{cases} \Rightarrow \begin{cases} \mathbb{I} \dot{\omega}_1 = -(\mathbb{I}_s - \mathbb{I}) \omega_2 \omega_3 & \text{(a)} \\ \mathbb{I} \dot{\omega}_2 = -(\mathbb{I} - \mathbb{I}_s) \omega_3 \omega_1 & \text{(b)} \\ \mathbb{I}_s \dot{\omega}_3 = 0 \end{cases}$$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ثابت} \quad \Omega = \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3$$

$$\text{(a)} \quad \dot{\omega}_1 + \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3 \omega_2 = 0 \Rightarrow \dot{\omega}_1 + \Omega \omega_2 = 0$$

$$\text{(b)} \quad \dot{\omega}_2 - \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3 \omega_1 = 0 \Rightarrow \dot{\omega}_2 - \Omega \omega_1 = 0$$

حرکت اجسام صلب در سه بعد

چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ثابت} \quad \Omega = \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3$$

$$\begin{cases} \dot{\omega}_1 + \Omega \omega_2 = 0 \\ \dot{\omega}_2 - \Omega \omega_1 = 0 \end{cases} \xrightarrow{\frac{d}{dt}} \begin{cases} \ddot{\omega}_1 + \Omega \dot{\omega}_2 = 0 \\ \ddot{\omega}_2 - \Omega \dot{\omega}_1 = 0 \end{cases}$$

$$\begin{cases} \ddot{\omega}_1 + \Omega \dot{\omega}_2 = 0 \\ \dot{\omega}_2 - \Omega \omega_1 = 0 \end{cases} \Rightarrow \ddot{\omega}_1 + \Omega^2 \omega_1 = 0 \Rightarrow \omega_1 = \omega_0 \cos(\Omega t + \delta) \Rightarrow \omega_1 = \omega_0 \cos \Omega t$$

$$\begin{cases} \dot{\omega}_1 + \Omega \omega_2 = 0 \\ \omega_1 = \omega_0 \cos \Omega t \end{cases} \Rightarrow \begin{cases} \omega_2 = -\frac{1}{\Omega} \dot{\omega}_1 \\ \omega_1 = \omega_0 \cos \Omega t \end{cases} \Rightarrow \omega_2 = \omega_0 \sin \Omega t$$

حرکت اجسام صلب در سه بعد

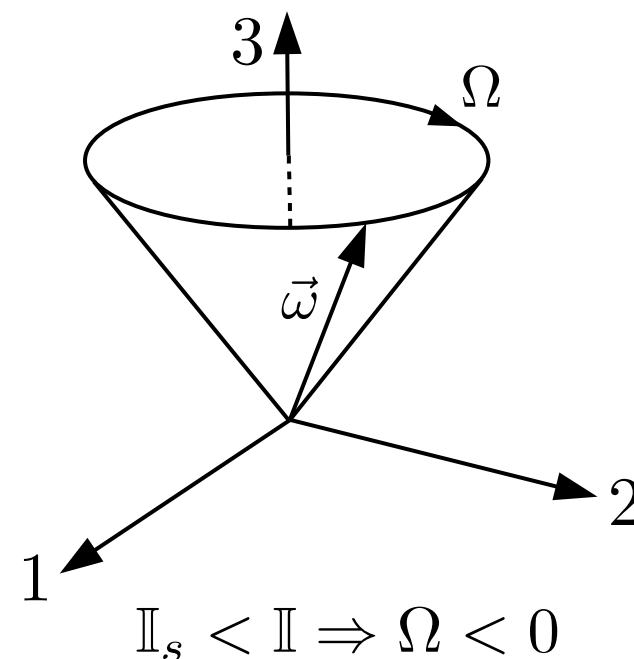
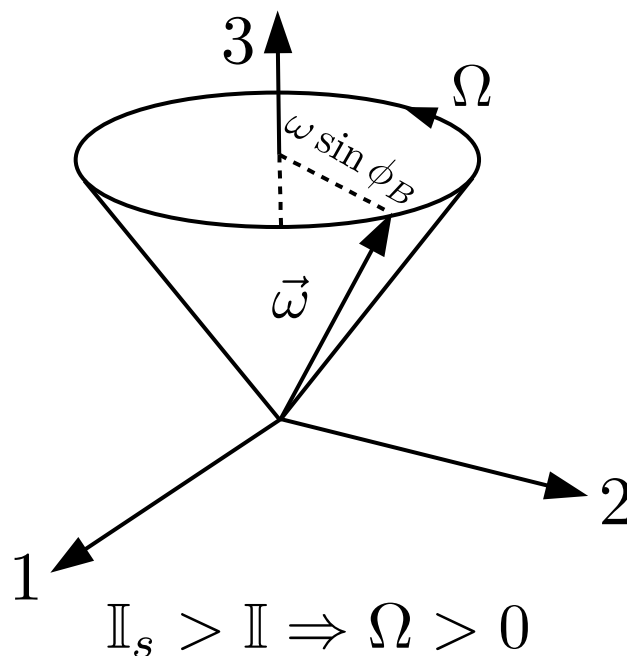
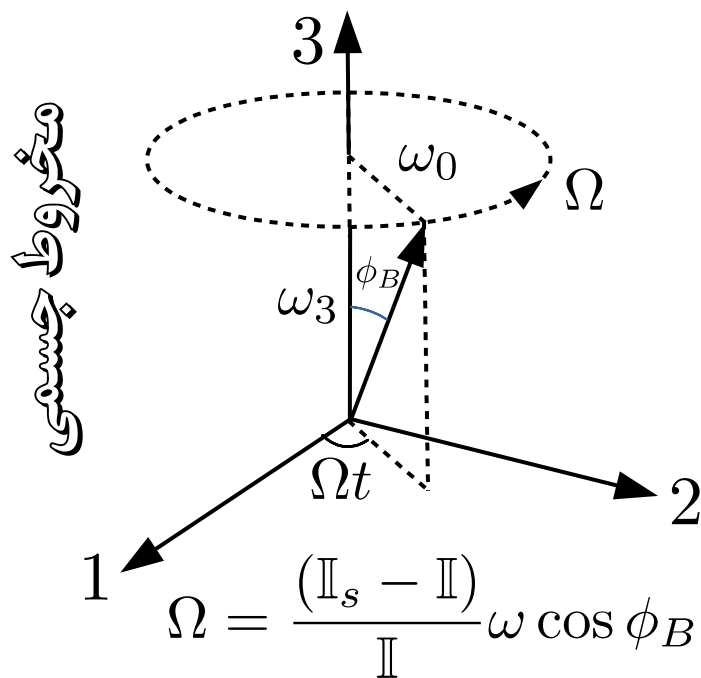
چرخش آزاد با یک محور تقارن

$$I = I_1 = I_2, \quad I_3 = I_s$$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ثابت}$$

$$\Omega = \frac{(I_s - I)}{I} \omega_3 \quad \begin{cases} \omega_1 = \omega_0 \cos \Omega t \\ \omega_2 = \omega_0 \sin \Omega t \end{cases}$$

$$\omega_3 = \omega \cos \phi_B$$



حرکت اجسام صلب در سه بعد

چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ثابت} \quad \Omega = \frac{(\mathbb{I}_s - \mathbb{I})}{\mathbb{I}} \omega_3 \quad \begin{cases} \omega_1 = \omega_0 \cos \Omega t \\ \omega_2 = \omega_0 \sin \Omega t \end{cases} \quad \omega_3 = \omega \cos \phi_B$$

مخروط فضایی

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega_0^2 + \omega_3^2} \Rightarrow \omega = \text{const.}$$

چرخش آزاد: $T_{\text{tot}} = \frac{1}{2} \mathbb{I}_1 \omega_1^2 + \frac{1}{2} \mathbb{I}_2 \omega_2^2 + \frac{1}{2} \mathbb{I}_3 \omega_3^2 = \text{const.}$

از طرفی می‌توان انرژی جنبشی دورانی را بصورت $2T_{\text{tot}} = \vec{L} \cdot \vec{\omega} \xrightarrow{T_{\text{tot}} = \text{const.}} \vec{L} \cdot \vec{\omega} = \text{const.}$

چرخش آزاد: $L^2 = \mathbb{I}_1^2 \omega_1^2 + \mathbb{I}_2^2 \omega_2^2 + \mathbb{I}_3^2 \omega_3^2 = \text{const.} \Rightarrow L = \text{const.}$

حرکت اجسام صلب در سه بعد

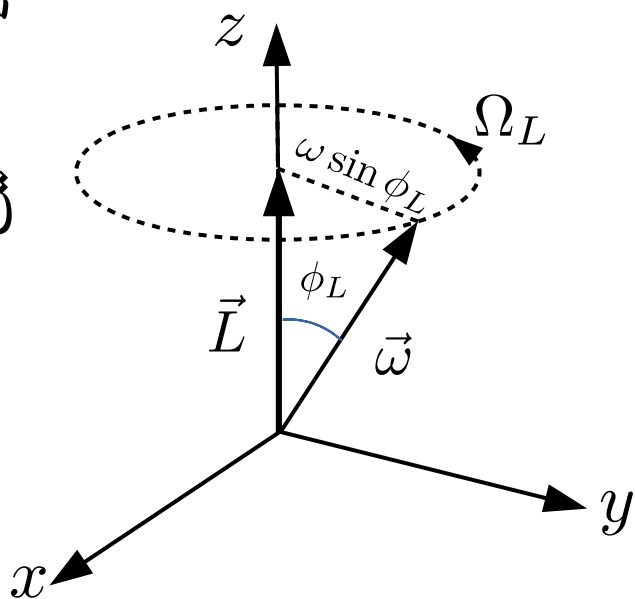
چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

$\omega = \text{const.} \quad \vec{L} \cdot \vec{\omega} = \text{const.} \quad L = \text{const.}$ دستگاه محورهاى اصلی (نالخت)



$\omega = \text{const.} \quad \vec{L} \cdot \vec{\omega} = \text{const.} \quad L = \text{const.}$ دستگاه مختصات ثابت (لخت)

مخروط فضایی



$$\cos \phi_L = \frac{\vec{\omega} \cdot \vec{L}}{\omega L}$$

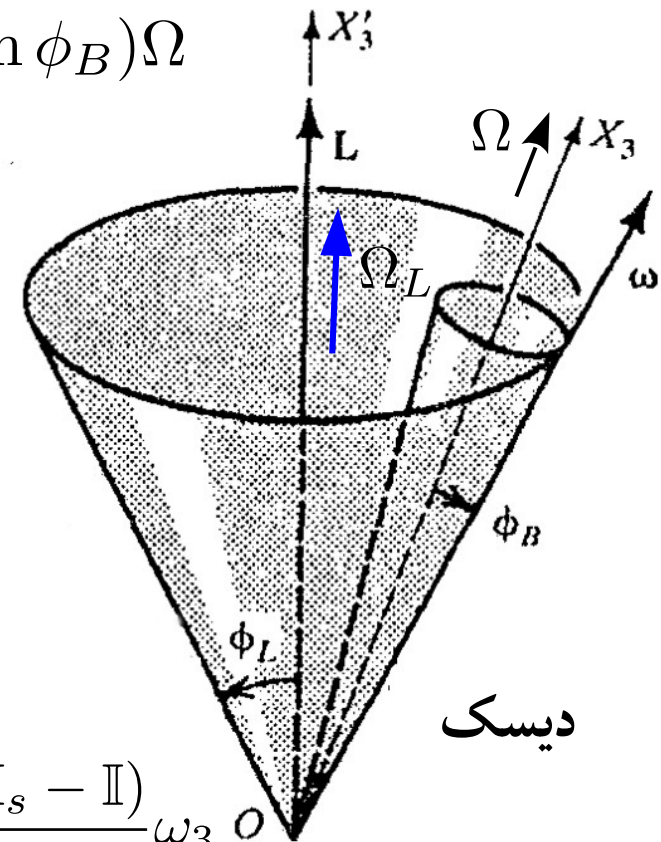
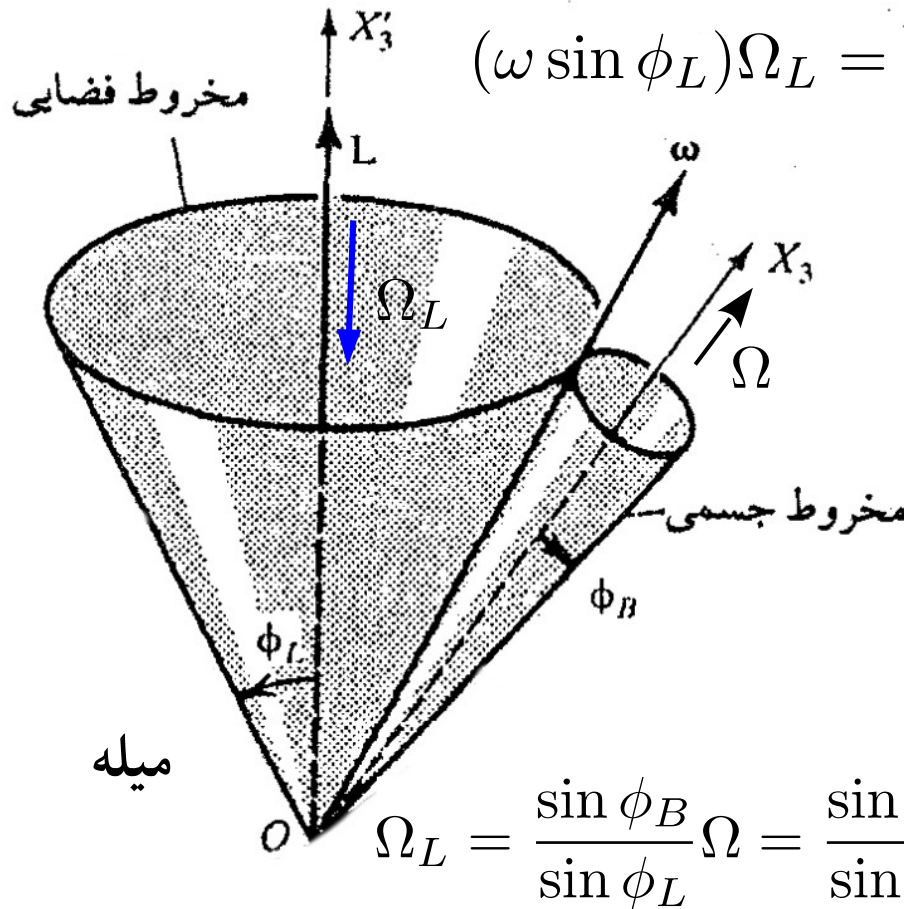
$$\cos \phi_L = \frac{2T_{tot}}{\omega L} = \text{const.}$$

حرکت اجسام صلب در سه بعد

چرخش آزاد با یک محور تقارن $\mathbb{I} = \mathbb{I}_1 = \mathbb{I}_2, \quad \mathbb{I}_3 = \mathbb{I}_s$

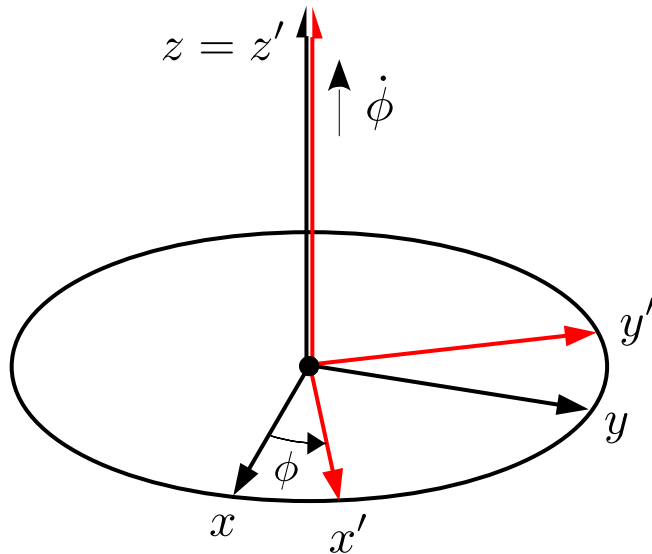
$$(\omega \sin \phi_L) \Omega_L = (\omega \sin \phi_B) \Omega$$

مخروط فضایی



حرکت اجسام صلب در سه بعد

زاویه تقدیمی، ϕ

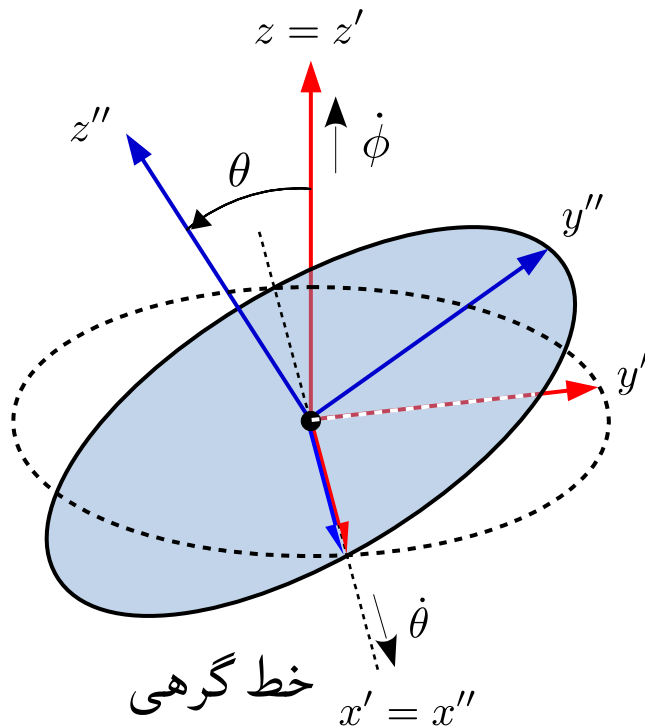


$$R_z(\phi) = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_z(\phi) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

حرکت اجسام صلب در سه بعد

زاویه ترقصی، θ

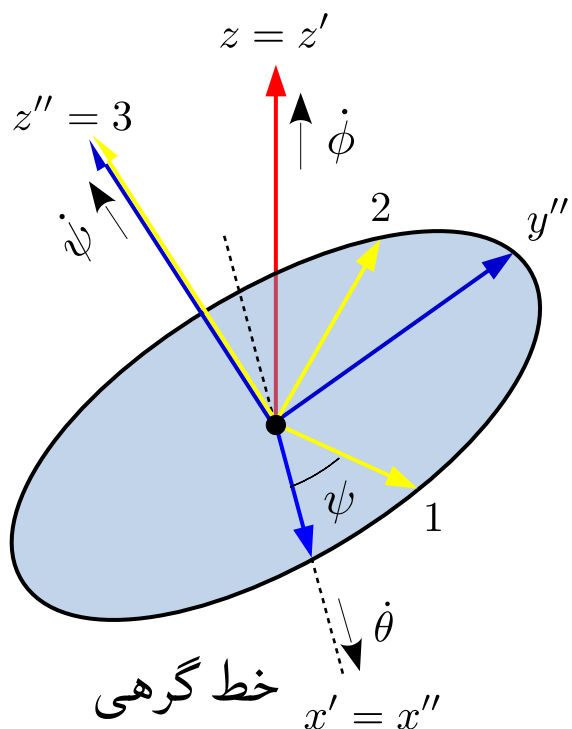


$$R_{x'}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = R_{x'}(\theta) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

حرکت اجسام صلب در سه بعد

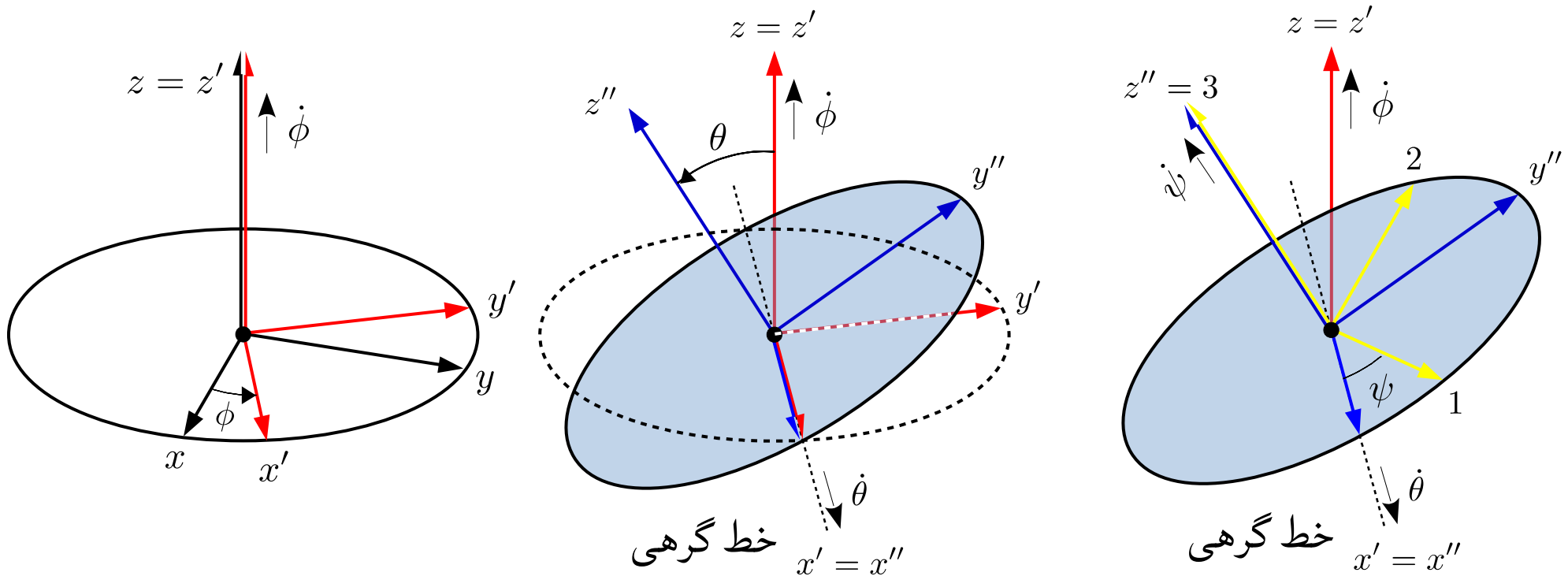
زاویه جسمی، ψ



$$R_{z''}(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

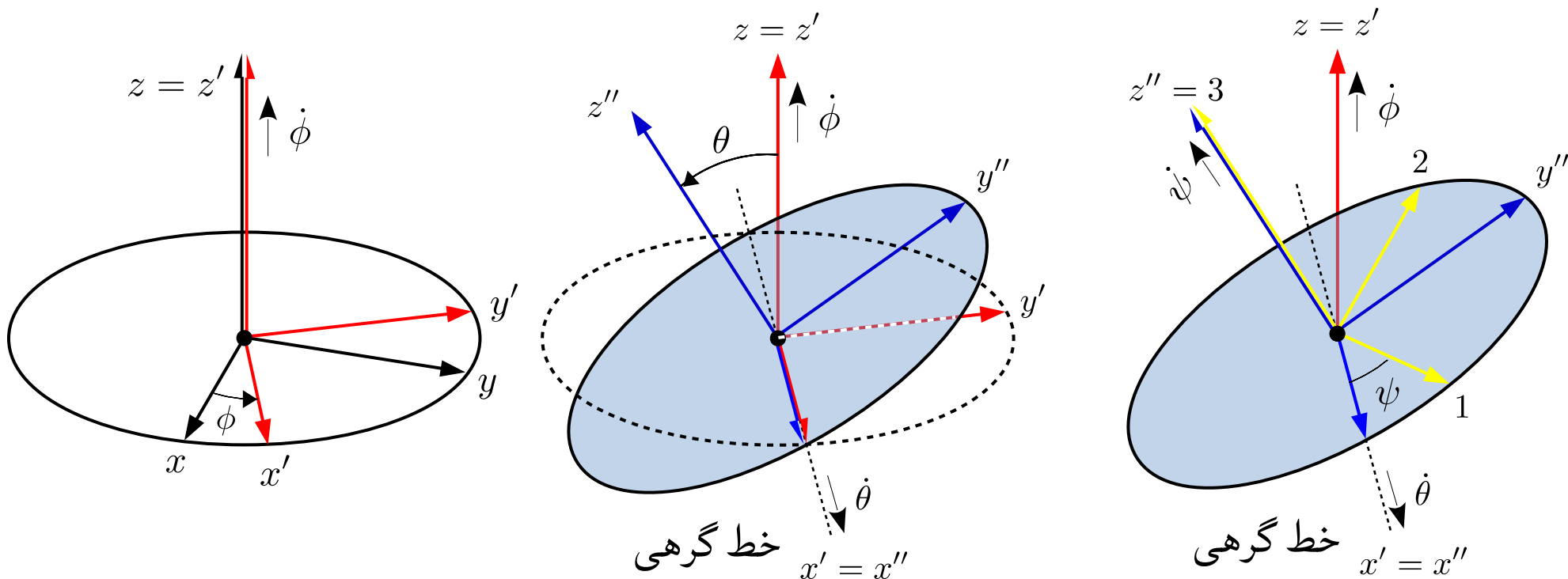
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = R_{z''}(\psi)(\theta) \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$

حرکت اجسام صلب در سه بعد



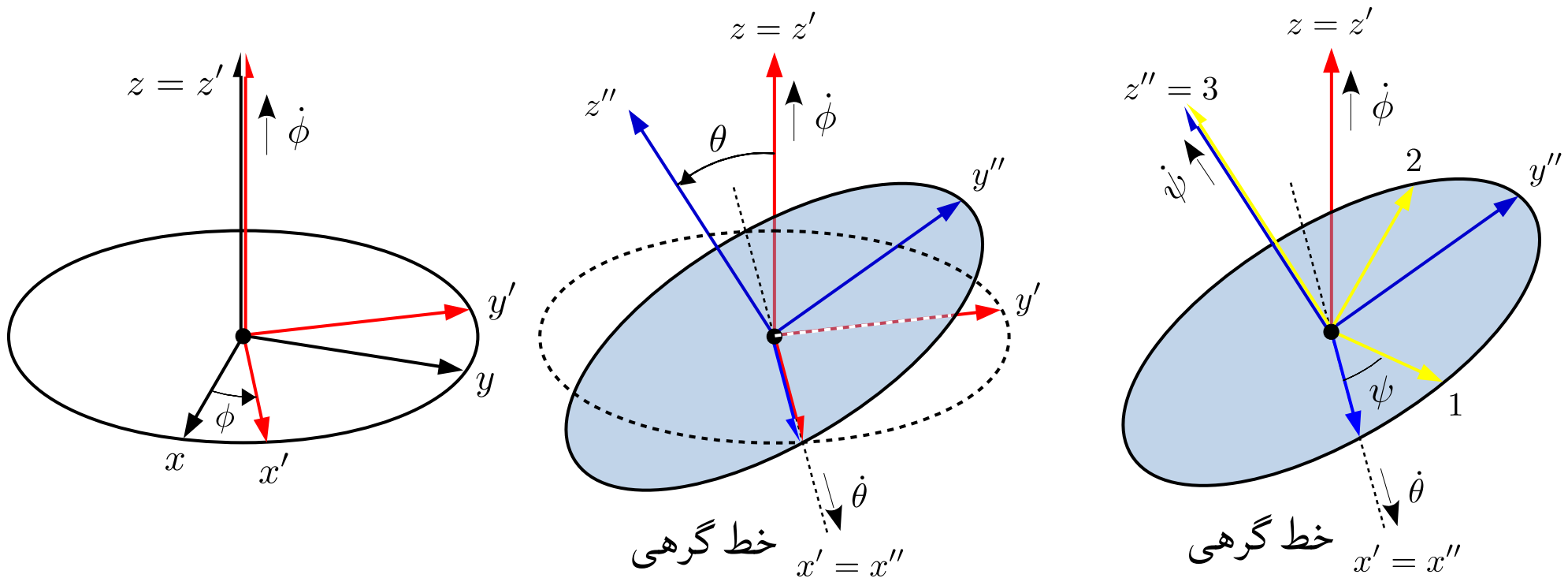
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = R_{z''}(\psi) R_{x'}(\theta) R_z(\phi) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

حرکت اجسام صلب در سه بعد



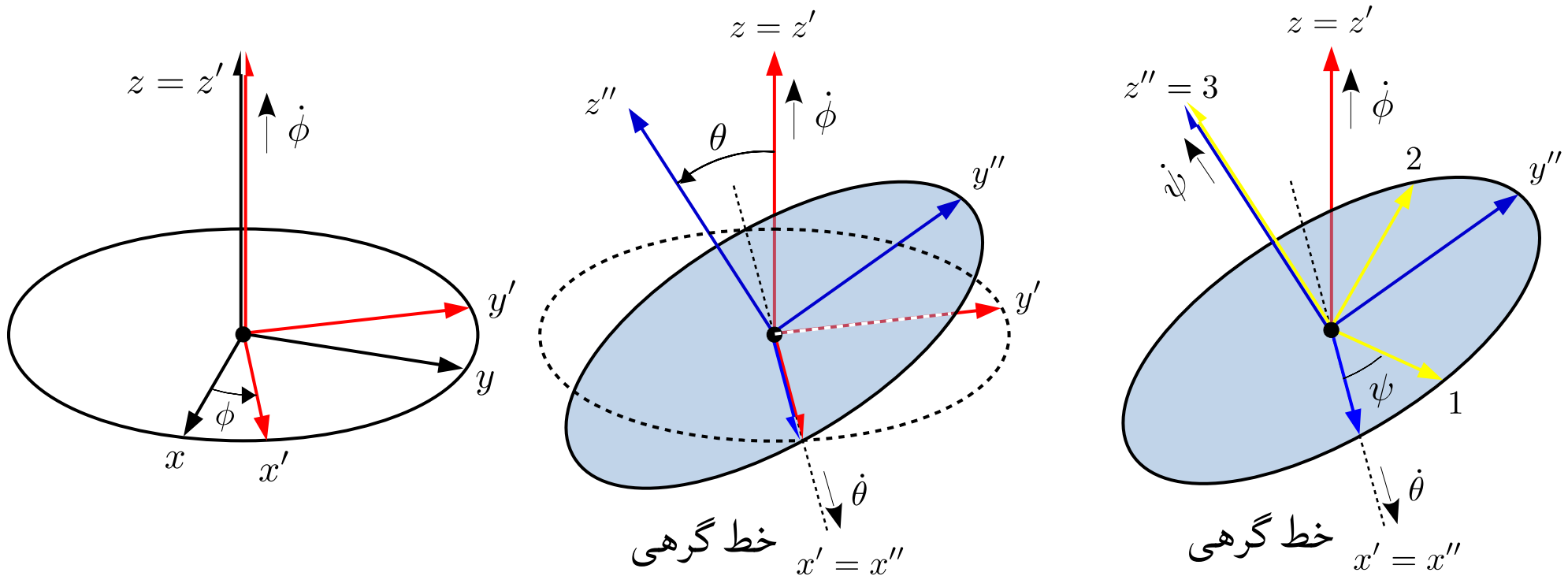
$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = R_{z''}(\psi) R_{x'}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = R_{z''}(\psi) \begin{pmatrix} 0 \\ \dot{\phi} \sin \theta \\ \dot{\phi} \cos \theta \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi \\ \dot{\phi} \sin \theta \cos \psi \\ \dot{\phi} \cos \theta \end{pmatrix}$$

حرکت اجسام صلب در سه بعد



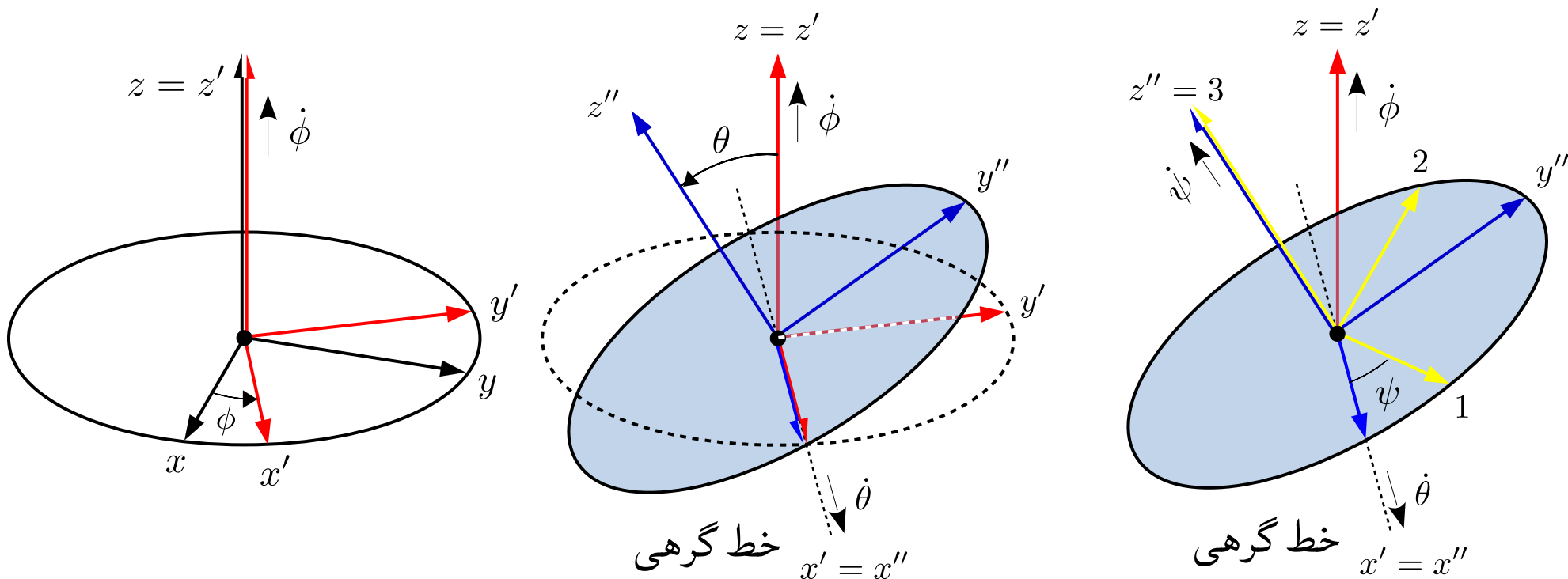
$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = R_{z''}(\psi) \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \cos \psi \\ -\dot{\theta} \sin \psi \\ 0 \end{pmatrix}$$

حرکت اجسام صلب در سه بعد



$$\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

حرکت اجسام صلب در سه بعد

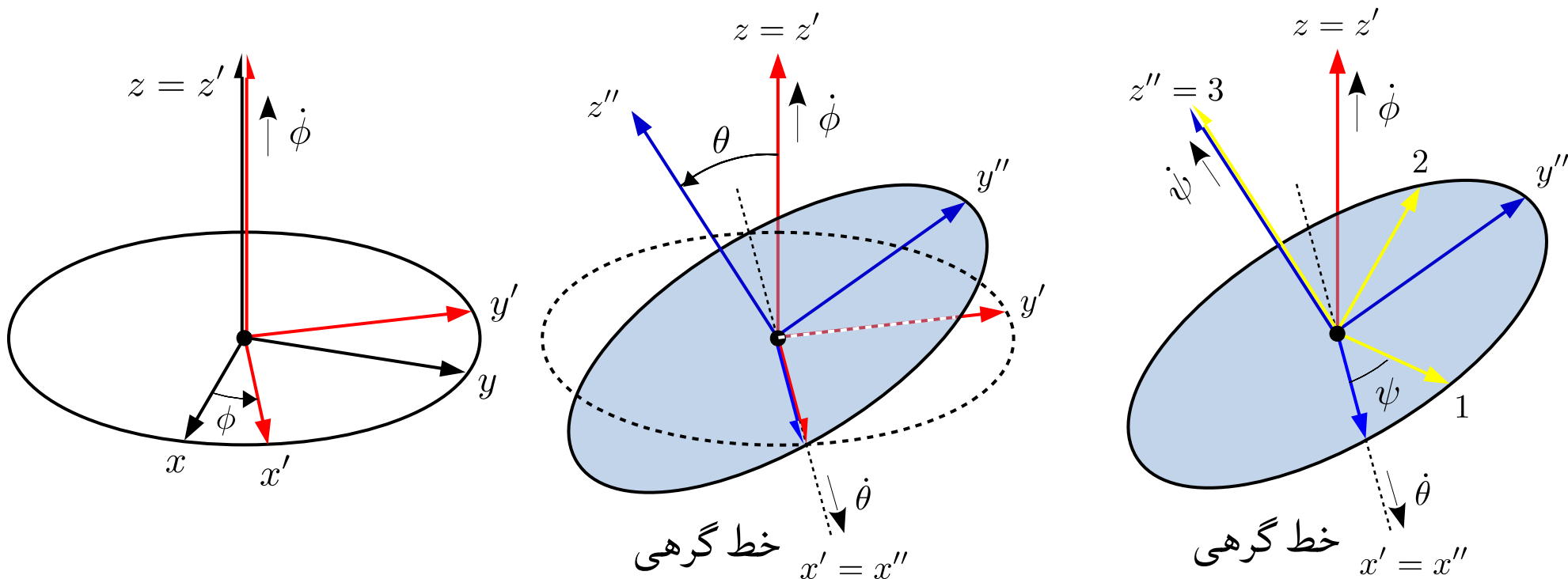


$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi \\ \dot{\phi} \sin \theta \cos \psi \\ \dot{\phi} \cos \theta \end{pmatrix}$$

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \cos \psi \\ -\dot{\theta} \sin \psi \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

حرکت اجسام صلب در سه بعد



$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} + \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{pmatrix}$$

حرکت اجسام صلب در سه بعد

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} + \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{pmatrix}$$

$$\begin{cases} \mathbb{I}_1 = \mathbb{I}_2 = \mathbb{I}, & \mathbb{I}_3 = \mathbb{I}_s \\ T = \frac{1}{2}\mathbb{I}_1\omega_1^2 + \frac{1}{2}\mathbb{I}_2\omega_2^2 + \frac{1}{2}\mathbb{I}_3\omega_3^2 \end{cases} \Rightarrow T = \frac{1}{2}\mathbb{I}(\omega_1^2 + \omega_2^2) + \frac{1}{2}\mathbb{I}_s\omega_3^2$$

$$T = \frac{1}{2}\mathbb{I}(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2}\mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi})^2, \quad V = 0$$

$$\mathcal{L} = T - V = \frac{1}{2}\mathbb{I}(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2}\mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \qquad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{\partial \mathcal{L}}{\partial \theta} \qquad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) = \frac{\partial \mathcal{L}}{\partial \psi}$$

حرکت اجسام صلب در سه بعد

$$\mathcal{L} = \frac{1}{2}\mathbb{I}(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2}\mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt} \left(\mathbb{I} \dot{\phi} \sin^2 \theta + \mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta \right) = 0 \Rightarrow \mathbb{I} \dot{\phi} \sin^2 \theta + \mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta = \text{const.}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad : \quad \mathbb{I} \dot{\phi} \sin^2 \theta + \mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta = p_\phi = \text{const.}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) = \frac{\partial \mathcal{L}}{\partial \psi}$$

$$\frac{d}{dt} \left(\mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) \right) = 0 \Rightarrow \mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) = \text{const.}$$

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \quad : \quad \mathbb{I}_s (\dot{\phi} \cos \theta + \dot{\psi}) = p_\psi = \text{const.}$$

حرکت اجسام صلب در سه بعد

$$\begin{cases} \mathbb{I}\dot{\phi} \sin^2 \theta + \mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi}) \cos \theta = p_\phi \\ \mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi}) = p_\psi \end{cases} \Rightarrow \begin{cases} \dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{\mathbb{I} \sin^2 \theta} \\ \dot{\psi} = \frac{p_\psi}{\mathbb{I}_s} - \frac{(p_\phi - p_\psi \cos \theta) \cos \theta}{\mathbb{I} \sin^2 \theta} \end{cases}$$

$$\mathcal{L} = \frac{1}{2}\mathbb{I}(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2}\mathbb{I}_s(\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$\mathcal{L} = \frac{1}{2}\mathbb{I}\dot{\theta}^2 + \frac{p_\psi^2}{2\mathbb{I}_s} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2\mathbb{I} \sin^2 \theta}, \quad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mathbb{I}\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{\mathbb{I}}$$

$$E = \frac{1}{2}\mathbb{I}\dot{\theta}^2 + \frac{p_\psi^2}{2\mathbb{I}_s} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2\mathbb{I} \sin^2 \theta}$$