Computational Physics

Lecture-08

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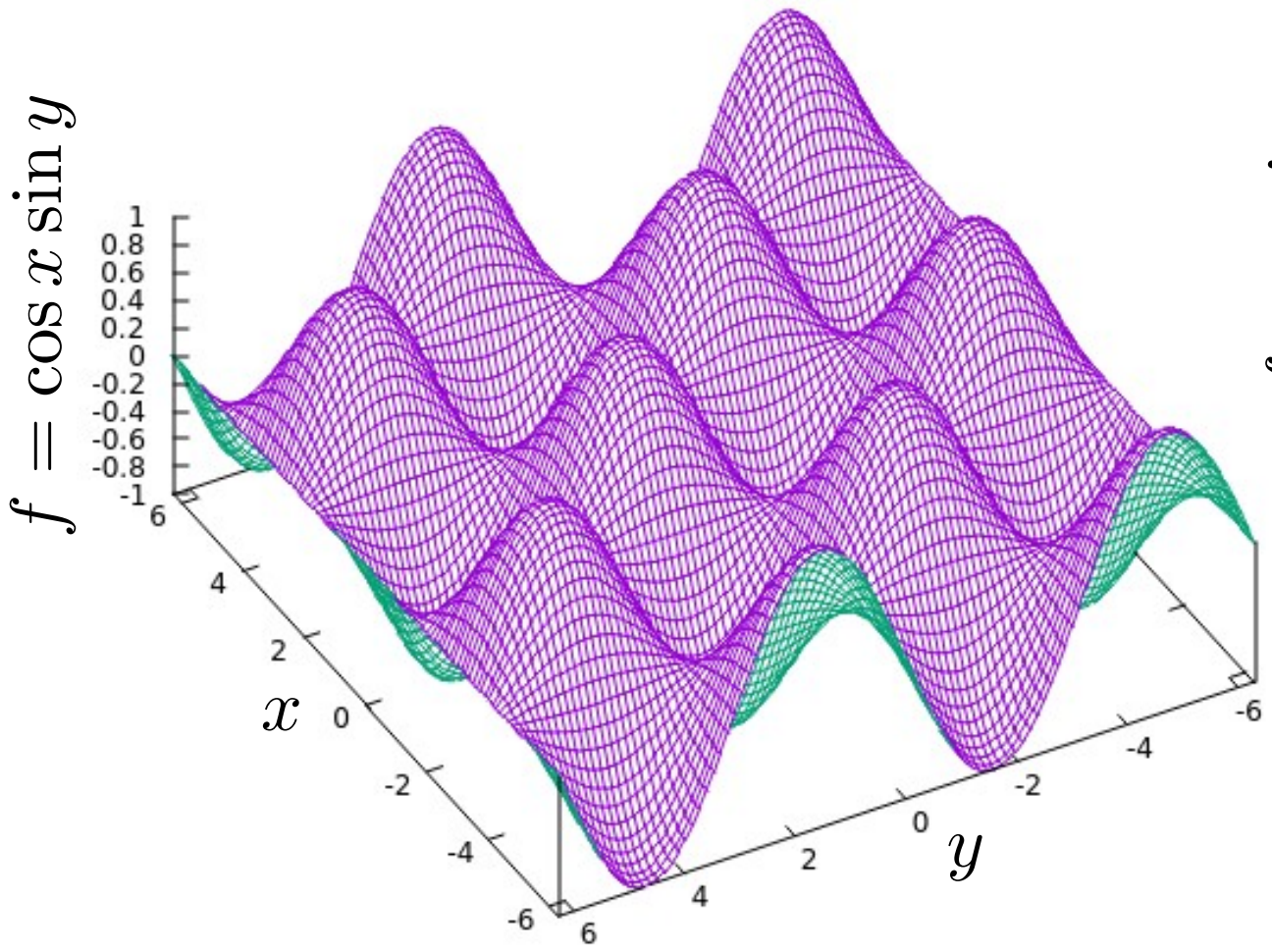
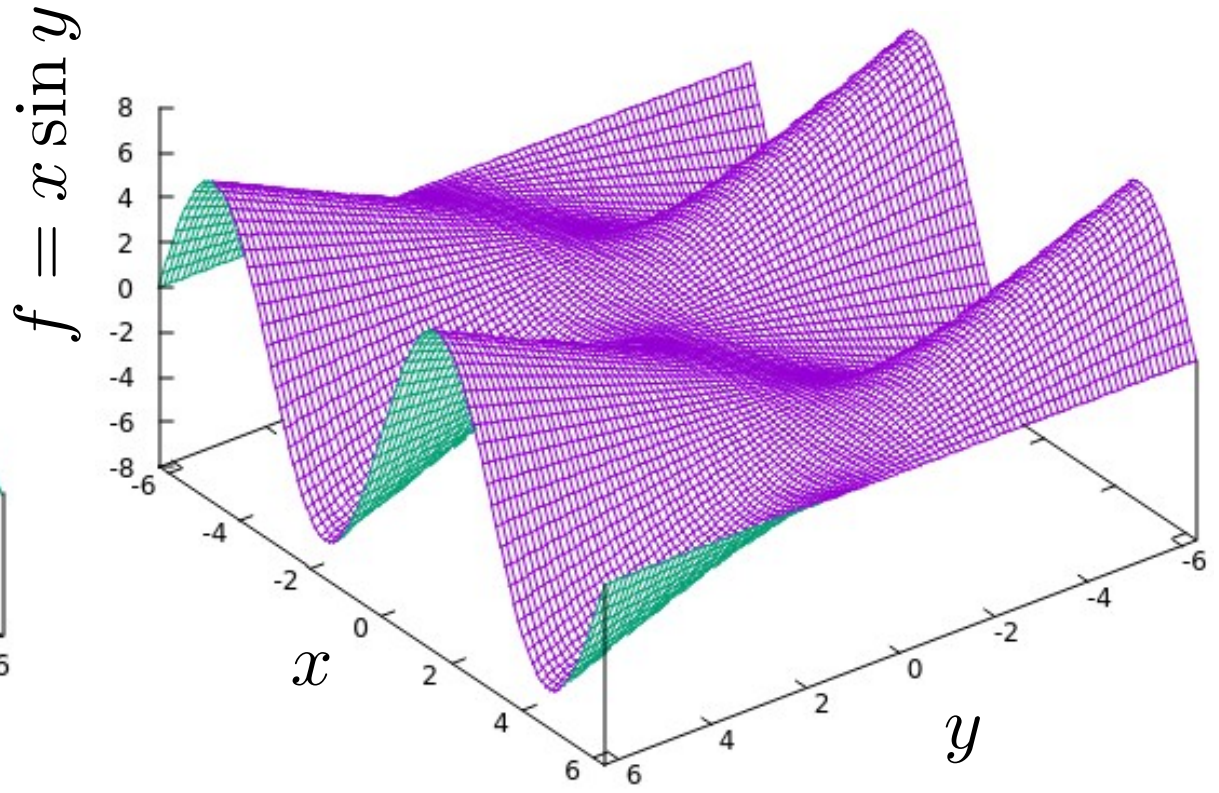
Numerical Integration

- Two-dimensional





Numerical Integration

- Two dimensional

$$f = f(x, y)$$

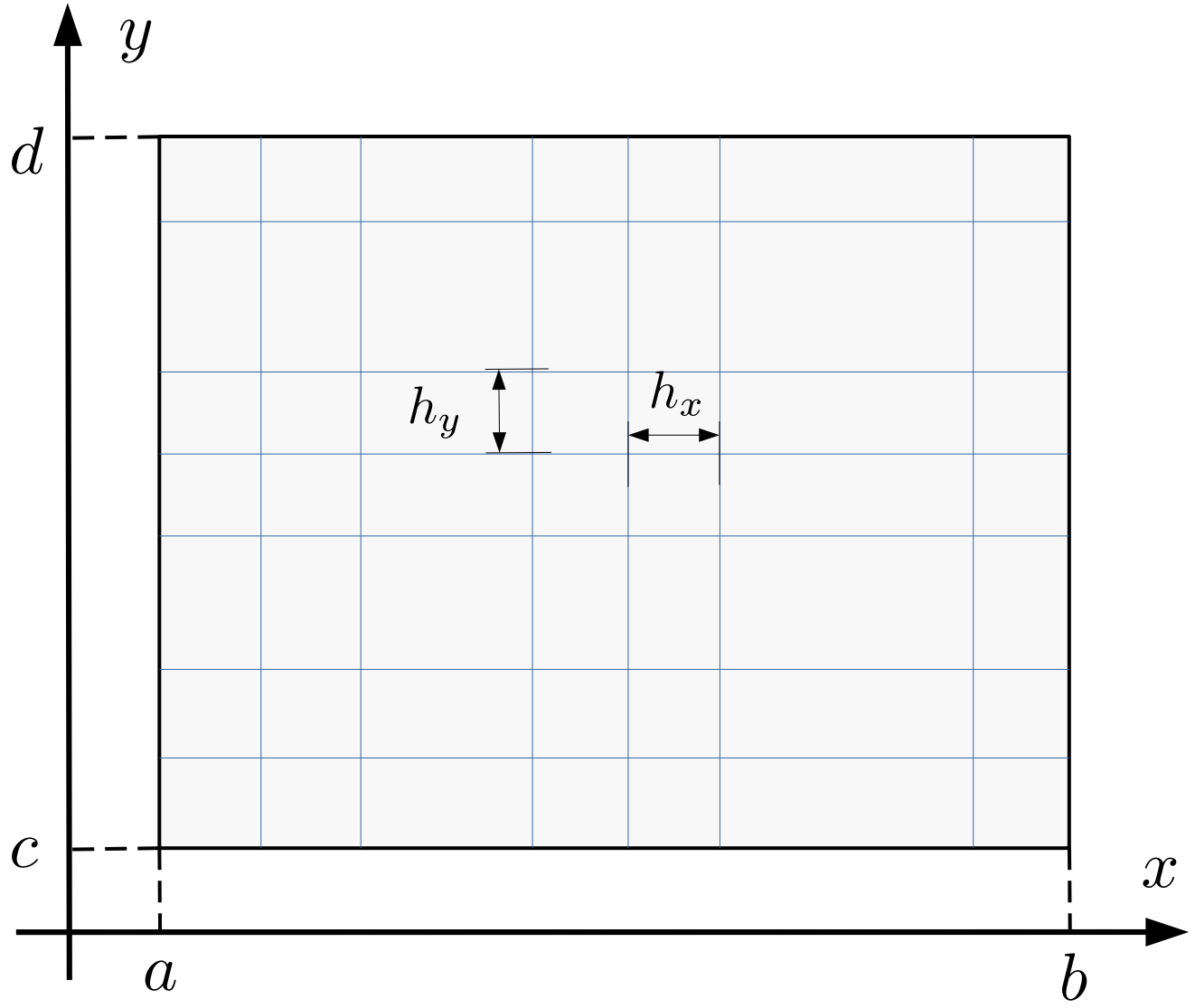
$$a \leq x \leq b$$

$$c \leq y \leq d$$

$$h_x = \frac{b-a}{M_x} \qquad h_y = \frac{d-c}{M_y}$$

$$x_i = a + ih_x \qquad y_j = c + jh_y$$

$$i = 0, 1, 2, \cdots, M_x \qquad j = 0, 1, 2, \cdots, M_y$$



Numerical Integration

- Two dimensional

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y$$

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = h_x \int_c^d \sum_{i=0}^{M_x} w_i^x f(x_i,y)\mathrm{d}y = h_x \sum_{i=0}^{M_x} w_i^x \left(h_y \sum_{j=0}^{M_y} w_j^y f(x_i,y_j) \right)$$

$$f(x_i,y_j) = f_{ij}$$

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = h_x h_y \sum_i^{M_x} \sum_j^{M_y} w_i^x w_j^y f_{ij} = h_x h_y \sum_{i=0}^{M_x} \sum_{j=0}^{M_y} w_i^x f_{ij} w_j^y$$

Numerical Integration

- Two dimensional

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = h_x h_y \sum_{i=0}^{M_x}\sum_{j=0}^{M_y} w_i^x f_{ij} w_j^y$$

□ Trapezoidal Rule $M_x = N_x, \quad M_y = N_y$

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = \frac{h_x}{2}\frac{h_y}{2} \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} w_i^x f_{ij} w_j^y$$

x	w^x	y	w^y
x_0	1	y_0	1
x_1	2	y_1	2
x_2	2	y_2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{i-1}	2	y_{j-1}	2
x_i	2	y_j	2
x_{i+1}	2	y_{j-1}	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{N_x-1}	2	y_{N_y-1}	2
x_{Nx}	1	y_{N_y}	1

Numerical Integration

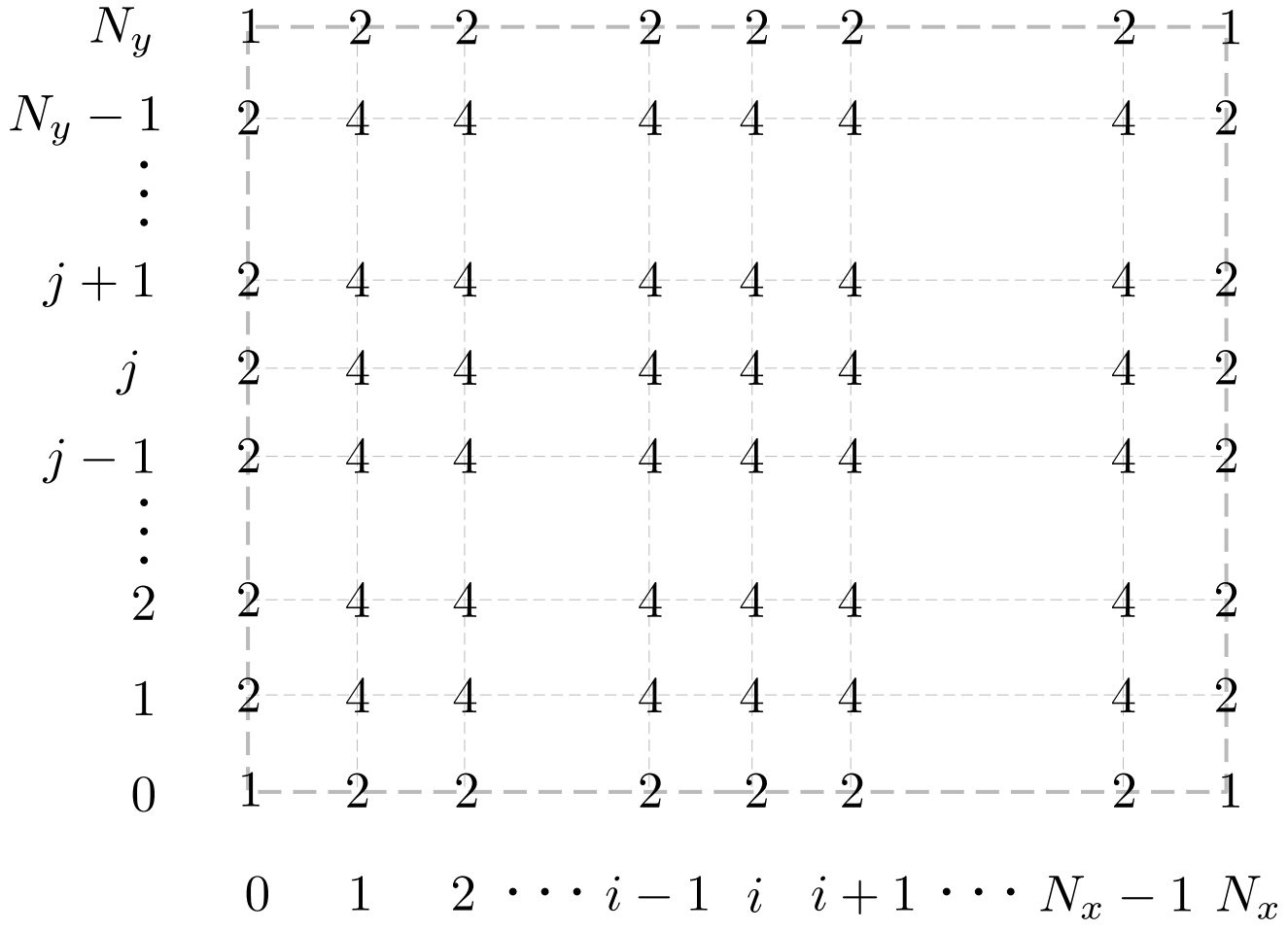
- Two dimensional
  - Trapezoidal Rule

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= \frac{h_x}{2}\frac{h_y}{2}\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} w_i^x f_{ij} w_j^y$$

$$= \frac{h_x}{2}\frac{h_y}{2}\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} w_{ij} f_{ij}$$

$$w_{ij} = w_i^x w_j^y$$



Numerical Integration

- Two dimensional

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = h_x h_y \sum_{i=0}^{M_x}\sum_{j=0}^{M_y} w_i^x f_{ij} w_j^y$$

□ Simpson's Rule $M_x = 2N_x, \quad M_y = 2N_y$

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = \frac{h_x}{3}\frac{h_y}{3} \sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y} w_i^x f_{ij} w_j^y$$

x	w^x	y	w^y
x_0	1	y_0	1
x_1	4	y_1	4
x_2	2	y_2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2i}	2	y_{2j}	2
x_{2i+1}	4	y_{2j+1}	4
x_{2i+2}	2	y_{2j+2}	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2N_x-1}	4	y_{2N_y-1}	4
x_{2N_x}	1	y_{2N_y}	1

Numerical Integration

- Two dimensional
  - □ Simpson's Rule

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= \frac{h_x}{3}\frac{h_y}{3}\sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y} w_i^x f_{ij} w_j^y$$

$$= \frac{h_x}{3}\frac{h_y}{3}\sum_{i=0}^{2N_x}\sum_{j=0}^{2N_y} w_{ij} f_{ij}$$

$$w_{ij} = w_i^x w_j^y$$

$2N_y$	1	4	2		2	4	2		4	1
$2N_y-1$	4	16	8		8	16	8		16	4
$\vdots$										
$2j+2$	2	8	4		4	8	4		8	2
$2j+1$	4	16	8		8	16	8		16	4
$2j$	2	8	4		4	8	4		8	2
$\vdots$										
2	2	8	4		4	8	4		8	2
1	4	16	8		8	16	8		16	4
0	1	4	2		2	4	2		4	1
	0	1	2		$2i$	$2i\times 1$	$2i\times 2$	$\cdots$	$2N_x-1$	$2N_x$

Numerical Integration

- Two dimensional

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = h_x h_y \sum_{i=0}^{M_x}\sum_{j=0}^{M_y} w_i^x f_{ij} w_j^y$$

- □ Simpson's 3/8-Rule $M_x = 3N_x, \quad M_y = 3N_y$

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y = \frac{3h_x}{8}\frac{3h_y}{8} \sum_{i=0}^{3N_x}\sum_{j=0}^{3N_y} w_i^x f_{ij} w_j^y$$

x	w^x	y	w^y
x_0	1	y_0	1
x_1	3	y_1	3
x_2	3	y_2	3
x_3	2	y_3	2
⋮	⋮	⋮	⋮
x_{3N_x-3}	2	y_{3N_y-3}	2
x_{3N_x-2}	3	y_{3N_y-2}	3
x_{3N_x-1}	3	y_{3N_y-1}	3
x_{3N_x}	1	y_{3N_y}	1

Numerical Integration

- Two dimensional
  - Simpson's 3/8-Rule

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\mathrm{d}y =$$

$$= \frac{3h_x}{8}\frac{3h_y}{8}\sum_{i=0}^{3N_x}\sum_{j=0}^{3N_y} w_i^x f_{ij} w_j^y$$

$$= \frac{3h_x}{8}\frac{3h_y}{8}\sum_{i=0}^{3N_x}\sum_{j=0}^{3N_y} w_{ij} f_{ij}$$

$$w_{ij} = w_i^x w_j^y$$

1	3	3	2	2	3	3	2	2	3	3	1
3	9	9	6	6	9	9	6	6	9	9	3
3	9	9	6	6	9	9	6	6	9	9	3
2	6	6	4	4	6	6	4	4	6	6	2
2	6	6	4	4	6	6	4	4	6	6	2
3	9	9	6	6	9	9	6	6	9	9	3
3	9	9	6	6	9	9	6	6	9	9	3
2	6	6	4	4	6	6	4	4	6	6	2
2	6	6	4	4	6	6	4	4	6	6	2
3	9	9	6	6	9	9	6	6	9	9	3
3	9	9	6	6	9	9	6	6	9	9	3
1	3	3	2	2	3	3	2	2	3	3	1