Computational Physics

Lecture-23

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Contents

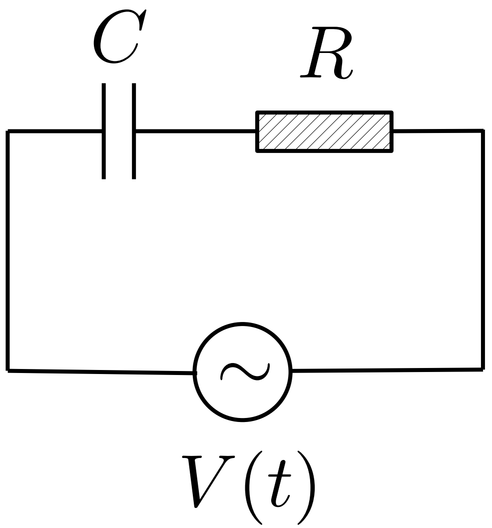
Ordinary Differential Equation

- Initial Value Problem

First Order ODE

$$\frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t} = V(t)$$

$$q(t = t_0) = q_0$$

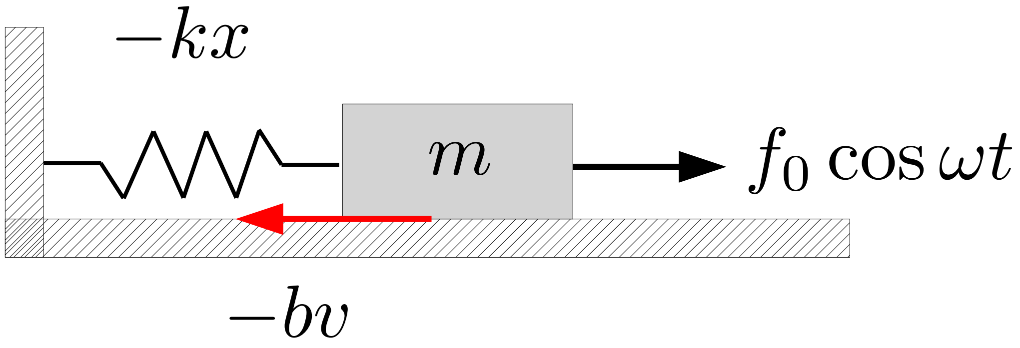


Second Order ODE

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - bv + f_0\cos\omega t$$

$$x(t = t_0) = x_0, \quad v(t = t_0) = v_0$$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$



• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

$$\dot{y} = f(y,t), \quad y(t=t_0) = y_0$$

First Order ODE

Second order Runge-Kutta

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

$$b_1,\ b_2,\ a_{12},\ c_2 = ?$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

$$\dot{y} = f(y,t), \quad y(t=t_0) = y_0$$

First Order ODE

Second order Runge-Kutta

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

• The elements a_{ij} form the **Runge-Kutta matrix** A ,

$$A = \begin{bmatrix} 0 & 0 \\ a_{12} & 0 \end{bmatrix}$$

• $c = (c_1, c_2)$ are the **Runge-Kutta nodes**, and for consistency each c_i must equal the sum of the ith row of A:

$$c_i = \sum_j a_{ij}$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

$$\dot{y} = f(y,t), \quad y(t=t_0) = y_0$$

First Order ODE

Second order Runge-Kutta

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

• $b = (b_1, b_2)$ are the **Runge-Kutta weights**.

• The Runge–Kutta matrix, nodes and weights are by convention written as a **Butcher tableau**:

$$\begin{array}{c|c} c & A \\ \hline & b \end{array}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c_2 & a_{12} & 0 \\ \hline & b_1 & b_2 \end{array}$$

• Ordinary Differential Equation (Initial Value Problem)

- Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + b_1 \Delta t f(y_n, t_n) + b_2 \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} k_1 f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + b_1 \Delta t f(y_n, t_n) + b_2 \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} \boxed{k_1} f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} \Delta t f(y_n, t_n) f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + b_1 \Delta t f(y_n, t_n) + b_2 \Delta t \boxed{f(y_n + a_{12} k_1, t_n + c_2 \Delta t)}$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} k_1 f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$\boxed{f(y_n + a_{12} k_1, t_n + c_2 \Delta t)} = f(y_n, t_n) + a_{12} \Delta t f(y_n, t_n) f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + b_1 \Delta t f(y_n, t_n) + b_2 \Delta t f(y_n + a_{12} k_1, t_n + c_2 \Delta t)$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} k_1 f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$f(y_n + a_{12} k_1, t_n + c_2 \Delta t) = f(y_n, t_n) + a_{12} \Delta t f(y_n, t_n) f_y(y_n, t_n) + c_2 \Delta t f_t(y_n, t_n)$$

$$y_{n+1} = y_n + (b_1 + b_2) \Delta t f(y_n, t_n) + b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

• Ordinary Differential Equation (Initial Value Problem)

- Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y(t_n + \Delta t) = y(t_n) + \Delta t \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_n + \frac{1}{2}\Delta t^2 \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)_n + \cdots$$

$$y_{n+1} = y_n + \Delta t \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_n + \frac{1}{2}\Delta t^2 \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)_n + \cdots$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_n + \frac{1}{2}\Delta t^2 \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)_n + \cdots$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t): \quad y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n + \cdots$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_n + \frac{1}{2}\Delta t^2 \left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right)_n + \cdots$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t): \quad y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \boxed{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n} + \cdots$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$

$$+b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \boxed{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n} + \cdots$$

$$f = f(y,t): \quad \mathrm{d}f = \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial f}{\partial t}\right)\mathrm{d}t = f_y \mathrm{d}y + f_t \mathrm{d}t$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$

$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \boxed{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n} + \cdots$$

$$f = f(y,t): \quad \mathrm{d}f = f_y \mathrm{d}y + f_t \mathrm{d}t \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = f_y \frac{\mathrm{d}y}{\mathrm{d}t} + f_t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y' = f(y,t)$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \boxed{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n} + \cdots$$

$$f = f(y,t): \quad \mathrm{d}f = f_y \mathrm{d}y + f_t \mathrm{d}t \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = f_y y' + f_t = f_y f + f_t$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$

$$+b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 \boxed{\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n} + \cdots$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = f_y f + f_t \Rightarrow \left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_n = f_y(y_n, t_n) f(y_n, t_n) + f_t(y_n, t_n)$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 [f_y(y_n, t_n) f(y_n, t_n) + f_t(y_n, t_n)] + \cdots$$

• Ordinary Differential Equation (Initial Value Problem)

- Runge-Kutta Method (Implicit Methods)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + (b_1 + b_2)\Delta t f(y_n, t_n)$$
$$+ b_2 \Delta t^2 (a_{12} f(y_n, t_n) f_y(y_n, t_n) + c_2 f_t(y_n, t_n))$$

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + \frac{1}{2}\Delta t^2 [f_y(y_n, t_n) f(y_n, t_n) + f_t(y_n, t_n)] + \cdots$$

$$b_1 + b_2 = 1, \quad b_2 a_{12} = \frac{1}{2}, \quad b_2 c_2 = \frac{1}{2} \Rightarrow b_1 = b_2 = \frac{1}{2}, \quad a_{12} = c_2 = 1$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

$$\dot{y} = f(y,t), \quad y(t=t_0) = y_0$$

First Order ODE

Second order Runge-Kutta

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = \Delta t f(y_n, t_n)$$

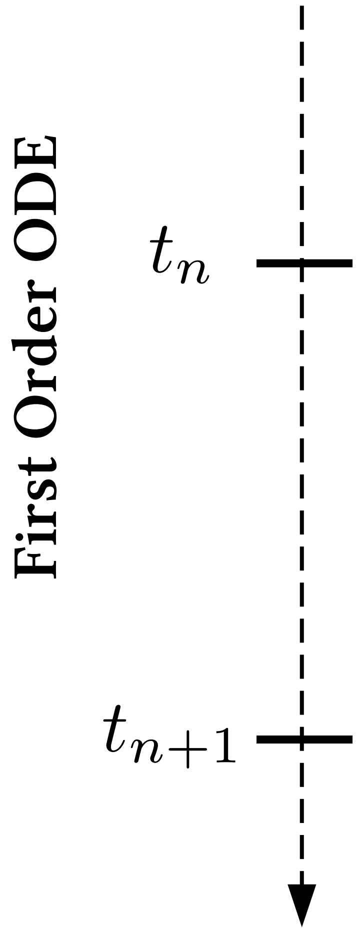
$$k_2 = \Delta t f(y_n + k_1, t_n + \Delta t)$$

Butcher tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)



Second order Runge-Kutta

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + k_1, t_n + \Delta t)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Stability of Runge-Kutta Method (Implicit Method)

First Order ODE

Second order Runge-Kutta

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + k_1, t_n + \Delta t)$$

$$y_{n+1} \to y_{n+1} + \delta y_{n+1}$$

$$y_n \to y_n + \delta y_n$$

• Ordinary Differential Equation (Initial Value Problem)

• Stability of Runge-Kutta Method (Implicit Method)

Second order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2),$$

$$y_{n+1} \to y_{n+1} + \delta y_{n+1}$$
$$y_n \to y_n + \delta y_n$$

$$\underbrace{y_{n+1} + \delta y_{n+1}}_{y_{n+1}} = \underbrace{y_n + \delta y_n}_{y_n} + \frac{1}{2}k_1(\underbrace{y_n + \delta y_n}_{y_n}, t_n)$$
$$+\frac{1}{2}k_2(\underbrace{y_n + \delta y_n}_{y_n} + k_1(\underbrace{y_n + \delta y_n}_{y_n}, t_n), t_n)$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Stability of Runge-Kutta Method (Implicit Method)

First Order ODE

Second order Runge-Kutta $\quad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

$$y_{n+1} + \delta y_{n+1} = y_n + \delta y_n + \frac{1}{2} k_1(y_n + \delta y_n, t_n)$$
$$+ \frac{1}{2} k_2(y_n + \delta y_n + k_1(y_n + \delta y_n, t_n), t_n)$$

$$k_1(y_n + \delta y_n, t_n) = k_1(y_n, t_n) + \left(\frac{\partial k_1}{\partial y_n}\right)_{t_n} \delta y_n + O(\delta y_n^2)$$

$$k_1(y_n + \delta y_n, t_n) = k_1(y_n, t_n) + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} \delta y_n + O(\delta y_n^2)$$

• Ordinary Differential Equation (Initial Value Problem)

- Stability of Runge-Kutta Method (Implicit Method)

First Order ODE

Second order Runge-Kutta $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

$$y_{n+1} + \delta y_{n+1} = y_n + \delta y_n + \frac{1}{2}k_1(y_n + \delta y_n, t_n)$$
$$+ \frac{1}{2}\boxed{k_2(y_n + \delta y_n + k_1(y_n + \delta y_n, t_n), t_n)}$$

$$\boxed{k_2(y_n + \delta y_n + k_1(y_n + \delta y_n, t_n), t_n)} = k_2\left(y_n + k_1 + \delta y_n \left[1 + \left(\frac{\partial k_1}{\partial y_n}\right)_{t_n}\right], t_n\right)$$

$$= k_2(y_n + k_1, t_n) + \delta y_n \left[1 + \left(\frac{\partial k_1}{\partial y_n}\right)_{t_n}\right] \left(\frac{\partial k_2}{\partial (y_n + k_1)}\right)_{t_n} + O(\delta y_n^2)$$

• Ordinary Differential Equation (Initial Value Problem)

• Stability of Runge-Kutta Method (Implicit Method)

Second order Runge-Kutta $$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

First Order ODE

$$y_{n+1} + \delta y_{n+1} = y_n + \delta y_n + \frac{1}{2} k_1(y_n + \delta y_n, t_n) + \frac{1}{2} \boxed{k_2(y_n + \delta y_n + k_1(y_n + \delta y_n, t_n), t_n)}$$

$$k_2\left(y_n + k_1 + \delta y_n \left[1 + \left(\frac{\partial k_1}{\partial y_n}\right)_{t_n}\right], t_n\right) =$$

$$= k_2(y_n + k_1, t_n) + \delta y_n \left[1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}\right] \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + O(\delta y_n^2)$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Stability of Runge-Kutta Method (Implicit Method)

Second order Runge-Kutta $$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

First Order ODE

$$y_{n+1} + \delta y_{n+1} = y_n + \delta y_n + \frac{1}{2}k_1(y_n, t_n) + \frac{1}{2}\Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} \delta y_n$$

$$+\frac{1}{2}k_2(y_n + k_1, t_n) + \frac{1}{2}\delta y_n \left[1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}\right] \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}$$

$$\delta y_{n+1} = \delta y_n + \frac{1}{2}\Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} \delta y_n + \frac{1}{2}\delta y_n \left[1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}\right] \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}$$

• Ordinary Differential Equation (Initial Value Problem)

• Stability of Runge-Kutta Method (Implicit Method)

Second order Runge-Kutta $$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

First Order ODE

$$\delta y_{n+1} = \delta y_n + \frac{1}{2}\Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} \delta y_n + \frac{1}{2}\delta y_n \left[1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}\right] \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}$$

$$\frac{\delta y_{n+1}}{\delta y_n} = 1 + \frac{1}{2}\Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\left[1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}\right] \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n}$$

$$\frac{\delta y_{n+1}}{\delta y_n} = 1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)_{t_n}^2$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Stability of Runge-Kutta Method (Implicit Method)

First Order ODE

Second order Runge-Kutta $$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

Growth Factor: $$g = \frac{\delta y_{n+1}}{\delta y_n} = 1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^2_{t_n}$$

$$\left|\frac{\delta y_{n+1}}{\delta y_n}\right| \leq 1 \Rightarrow \boxed{\left|1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^2_{t_n}\right| \leq 1}$$

Stability Condition

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)

$$\dot{y} = f(y, t), \quad y(t = t_0) = y_0$$

Fourth order Runge-Kutta

First Order ODE

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t)$$

$$k_3 = \Delta t f(y_n + \frac{1}{2}k_2, t_n + \frac{1}{2}\Delta t)$$

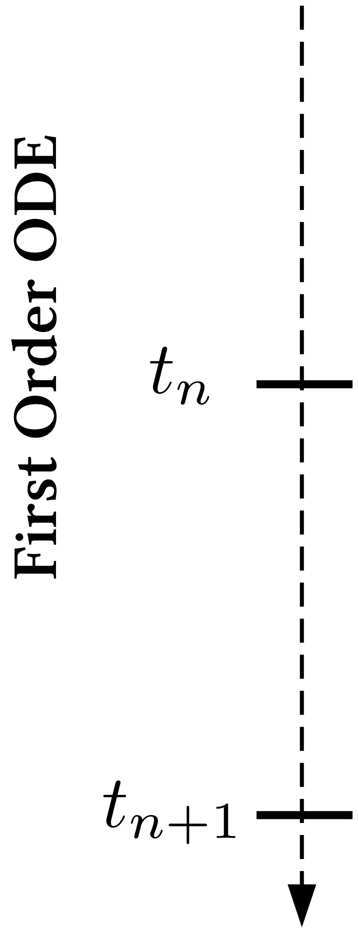
$$k_4 = \Delta t f(y_n + k_3, t_n + \Delta t)$$

Butcher tableau:

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Method (Implicit Methods)



Fourth order Runge-Kutta

$$k_1 = \Delta t f(y_n, t_n)$$

$$k_2 = \Delta t f(y_n + \frac{1}{2}k_1, t_n + \frac{1}{2}\Delta t)$$

$$k_3 = \Delta t f(y_n + \frac{1}{2}k_2, t_n + \frac{1}{2}\Delta t)$$

$$k_4 = \Delta t f(y_n + k_3, t_n + \Delta t)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Ordinary Differential Equation (Initial Value Problem)

- Stability of Runge-Kutta Method (Implicit Method)

First Order ODE

Second order Runge-Kutta $$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

Growth Factor:

$$\frac{\delta y_{n+1}}{\delta y_n} = 1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^2_{t_n} + \frac{1}{6}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^3_{t_n} + \frac{1}{24}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^4_{t_n}$$

Stability Condition $$\left|\frac{\delta y_{n+1}}{\delta y_n}\right| \leq 1$$

$$\left|1 + \Delta t \left(\frac{\partial f}{\partial y_n}\right)_{t_n} + \frac{1}{2}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^2_{t_n} + \frac{1}{6}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^3_{t_n} + \frac{1}{24}\Delta t^2 \left(\frac{\partial f}{\partial y_n}\right)^4_{t_n}\right| \leq 1$$