Computational Physics

Lecture-24

M. Reza Mozaffari

Physics Group, University of Qom

Contents

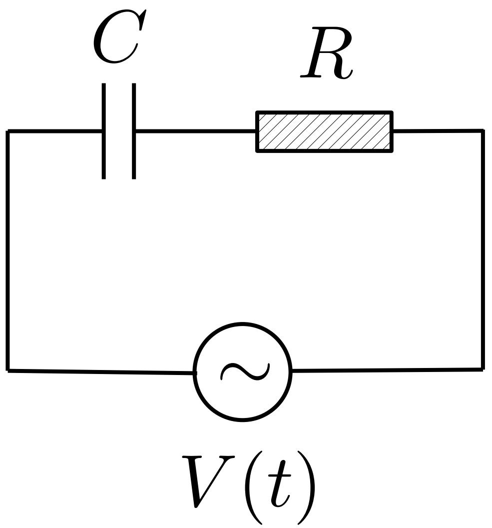
Ordinary Differential Equation

- Initial Value Problem

First Order ODE

$$\frac{q}{C} + R\frac{\mathrm{d}q}{\mathrm{d}t} = V(t)$$

$$q(t = t_0) = q_0$$

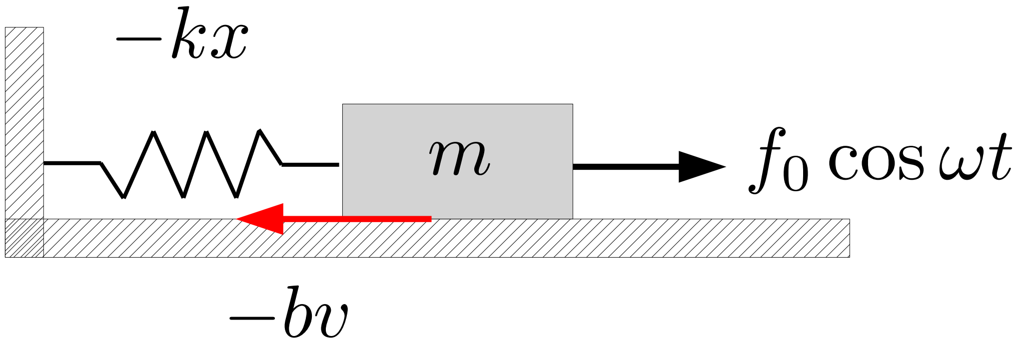


Second Order ODE

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - bv + f_0 \cos \omega t$$

$$x(t = t_0) = x_0, \quad v(t = t_0) = v_0$$

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$



• Ordinary Differential Equation (Initial Value Problem)

Second Order ODE

$$\ddot{y} = f(\dot{y}, y, t), \quad \dot{y}(t = t_0) = \dot{y}_0, \quad y(t = t_0) = y_0$$

$$\dot{y} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v$$

$$v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$

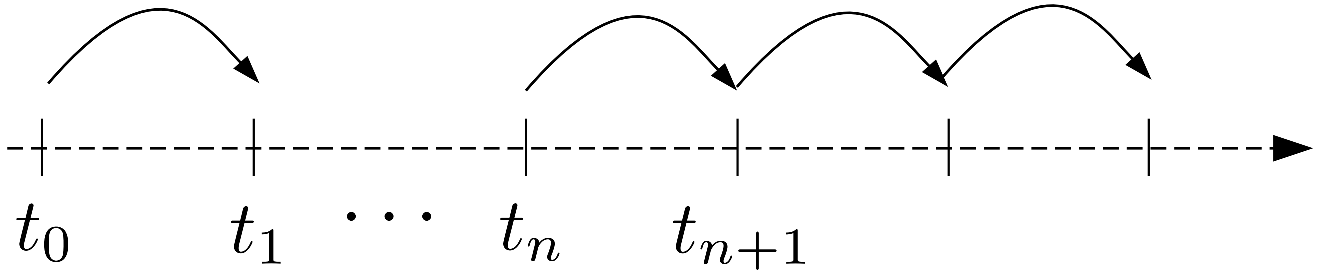
$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v \end{cases} \qquad \begin{cases} v(t = t_0) = v_0 \\ y(t = t_0) = y_0 \end{cases}$$

• Ordinary Differential Equation (Initial Value Problem)

Second Order ODE

• Euler Method (Explicit Method)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$



$$t_n = t_0 + n\Delta t$$

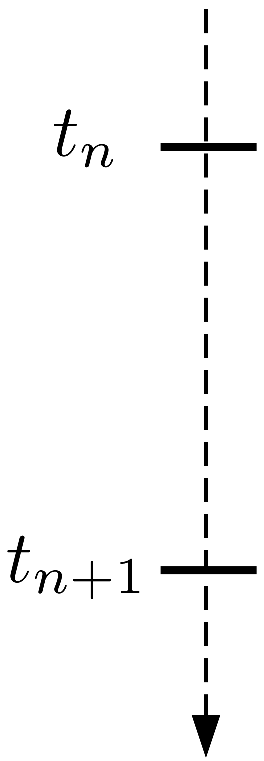
$$\begin{cases} v_{n+1} = v_n + f_n \Delta t + O(\Delta t^2), & f_n = f(v_n, y_n, t_n) \\ y_{n+1} = y_n + g_n \Delta t + O(\Delta t^2), & g_n = g(v_n, y_n, t_n) \end{cases}$$

• Ordinary Differential Equation (Initial Value Problem)

• Euler Method (Explicit Method)

Second Order ODE

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$



$$f_n = f(v_n, y_n, t_n)$$

$$g_n = g(v_n, y_n, t_n)$$

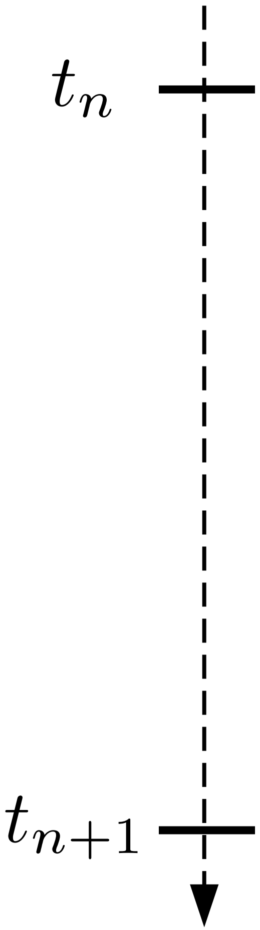
$$v_{n+1} = v_n + f_n \Delta t$$

$$y_{n+1} = y_n + g_n \Delta t$$

• Ordinary Differential Equation (Initial Value Problem)

• Picard Method (Implicit Methods)

Second Order ODE



One time

$$\tilde{v}_{n+1} = v_n + f(v_n, y_n, t_n)\Delta t$$

$$\tilde{y}_{n+1} = y_n + g(v_n, y_n, t_n)\Delta t$$

Euler Method

$$v_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$y_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Predictor–corrector methods (Implicit Methods)

Second Order ODE

t_n

Two times

$$\tilde{v}_{n+1} = v_n + f(v_n, y_n, t_n)\Delta t$$

$$\tilde{y}_{n+1} = y_n + g(v_n, y_n, t_n)\Delta t$$

Euler Method

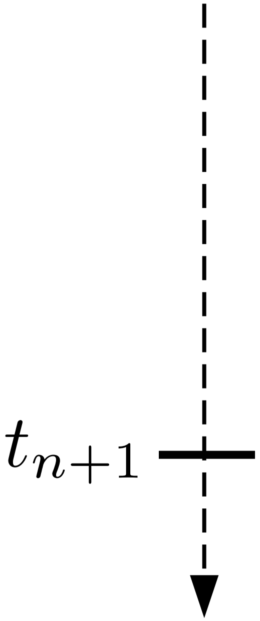
$$\hat{v}_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$\hat{y}_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Predictor–corrector methods (Implicit Methods)

Second Order ODE



Two times

$$v_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\hat{v}_{n+1}, \hat{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$y_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\hat{v}_{n+1}, \hat{y}_{n+1}, t_{n+1})\right)\Delta t$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Predictor–corrector methods (Implicit Methods)

Second Order ODE

t_n

Three times

$$\tilde{v}_{n+1} = v_n + f(v_n, y_n, t_n)\Delta t$$

$$\tilde{y}_{n+1} = y_n + g(v_n, y_n, t_n)\Delta t$$

Euler Method

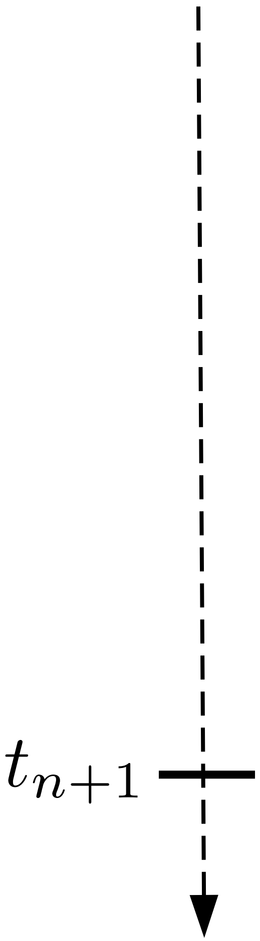
$$\hat{v}_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$\hat{y}_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\tilde{v}_{n+1}, \tilde{y}_{n+1}, t_{n+1})\right)\Delta t$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Predictor–corrector methods (Implicit Methods)

Second Order ODE



Three times

$$\bar{v}_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\hat{v}_{n+1}, \hat{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$\bar{y}_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\hat{v}_{n+1}, \hat{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$v_{n+1} = v_n + \frac{1}{2}\left(f(v_n, y_n, t_n) + f(\bar{v}_{n+1}, \bar{y}_{n+1}, t_{n+1})\right)\Delta t$$

$$y_{n+1} = y_n + \frac{1}{2}\left(g(v_n, y_n, t_n) + g(\bar{v}_{n+1}, \bar{y}_{n+1}, t_{n+1})\right)\Delta t$$

• Ordinary Differential Equation (Initial Value Problem)

• Runge-Kutta Methods (Implicit Methods)

Second Order ODE

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$

t_n

$$k_1 = \Delta t f(v_n, y_n, t_n)$$
$$q_1 = \Delta t g(v_n, y_n, t_n)$$

$$k_2 = \Delta t f(v_n + k_1, y_n + q_1, t_n + \Delta t)$$
$$q_2 = \Delta t g(v_n + k_1, y_n + q_1, t_n + \Delta t)$$

$$y_{n+1} = y_n + \frac{1}{2}(q_1 + q_2)$$
$$v_{n+1} = v_n + \frac{1}{2}(k_1 + k_2)$$

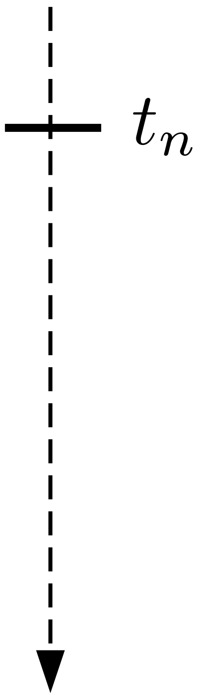
t_{n+1}

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Methods (Implicit Methods)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$

Second Order ODE



$$k_1 = \Delta t f(v_n, y_n, t_n)$$
$$q_1 = \Delta t g(v_n, y_n, t_n)$$

$$k_2 = \Delta t f(v_n + \frac{1}{2}k_1, y_n + \frac{1}{2}q_1, t_n + \frac{1}{2}\Delta t)$$
$$q_2 = \Delta t g(v_n + \frac{1}{2}k_1, y_n + \frac{1}{2}q_1, t_n + \frac{1}{2}\Delta t)$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Methods (Implicit Methods)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$

Second Order ODE

$$k_3 = \Delta t f(v_n + \frac{1}{2}k_2, y_n + \frac{1}{2}q_2, t_n + \frac{1}{2}\Delta t)$$

$$q_3 = \Delta t g(v_n + \frac{1}{2}k_2, y_n + \frac{1}{2}q_2, t_n + \frac{1}{2}\Delta t)$$

$$k_4 = \Delta t f(v_n + k_3, y_n + q_3, t_n + \Delta t)$$

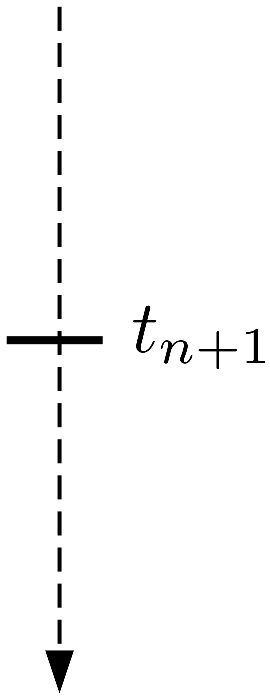
$$q_4 = \Delta t g(v_n + k_3, y_n + q_3, t_n + \Delta t)$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Runge-Kutta Methods (Implicit Methods)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(v, y, t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(v, y, t) = v, \quad v(t = t_0) = v_0, \quad y(t = t_0) = y_0$$

Second Order ODE

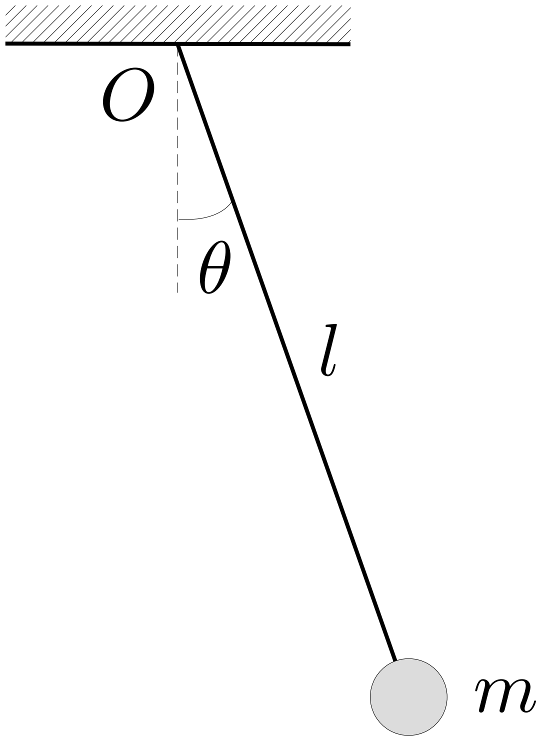


$$y_{n+1} = y_n + \frac{1}{6}(q_1 + 2q_2 + 2q_3 + q_4)$$

$$v_{n+1} = v_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

• **Ordinary Differential Equation** (Initial Value Problem)

- Driven Pendulum



$$\tau = I_O \ddot{\theta}$$

$$ml^2 \ddot{\theta} = -mgl \sin\theta - b\dot{\theta} + \tau_0 \cos\omega_{\mathrm{d}} t$$

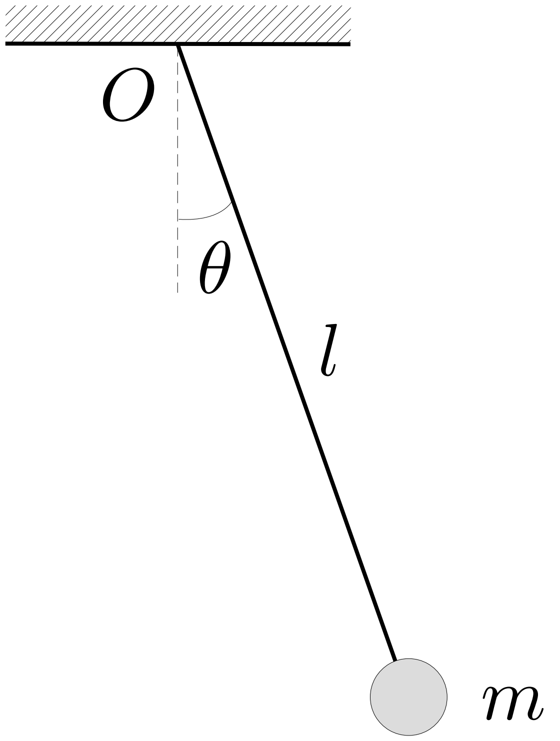
$$\frac{ml^2}{ml^2}\ddot{\theta} = -\frac{mgl}{ml^2}\sin\theta - \frac{b}{ml^2}\dot{\theta} + \frac{\tau_0}{ml^2}\cos\omega_{\mathrm{d}} t$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{b}{ml^2}\dot{\theta} + \frac{\tau_0}{ml^2}\cos\omega_{\mathrm{d}} t$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

• **Ordinary Differential Equation** (Initial Value Problem)

• Driven Pendulum



$$\ddot{\theta} = -\omega_0^2 \sin\theta - \frac{b}{ml^2}\dot{\theta} + \frac{\tau_0}{ml^2}\cos\omega_{\mathrm{d}}t$$

$$\begin{cases} \dot{\omega} = f(\omega, \theta, t) = -\omega_0^2 \sin\theta - \dfrac{b}{ml^2}\omega + \dfrac{\tau_0}{ml^2}\cos\omega_{\mathrm{d}}t \\ \dot{\theta} = g(\omega, \theta, t) = \omega \end{cases}$$

ω_0	$\frac{b}{ml^2}$	ω_{d}
1	$1/2$	$2/3$

$$\begin{cases} \dfrac{\tau_0}{ml^2} = 0.9 \\ \dfrac{\tau_0}{ml^2} = 1.15 \end{cases}$$

• Ordinary Differential Equation (Initial Value Problem)

• Driven Pendulum

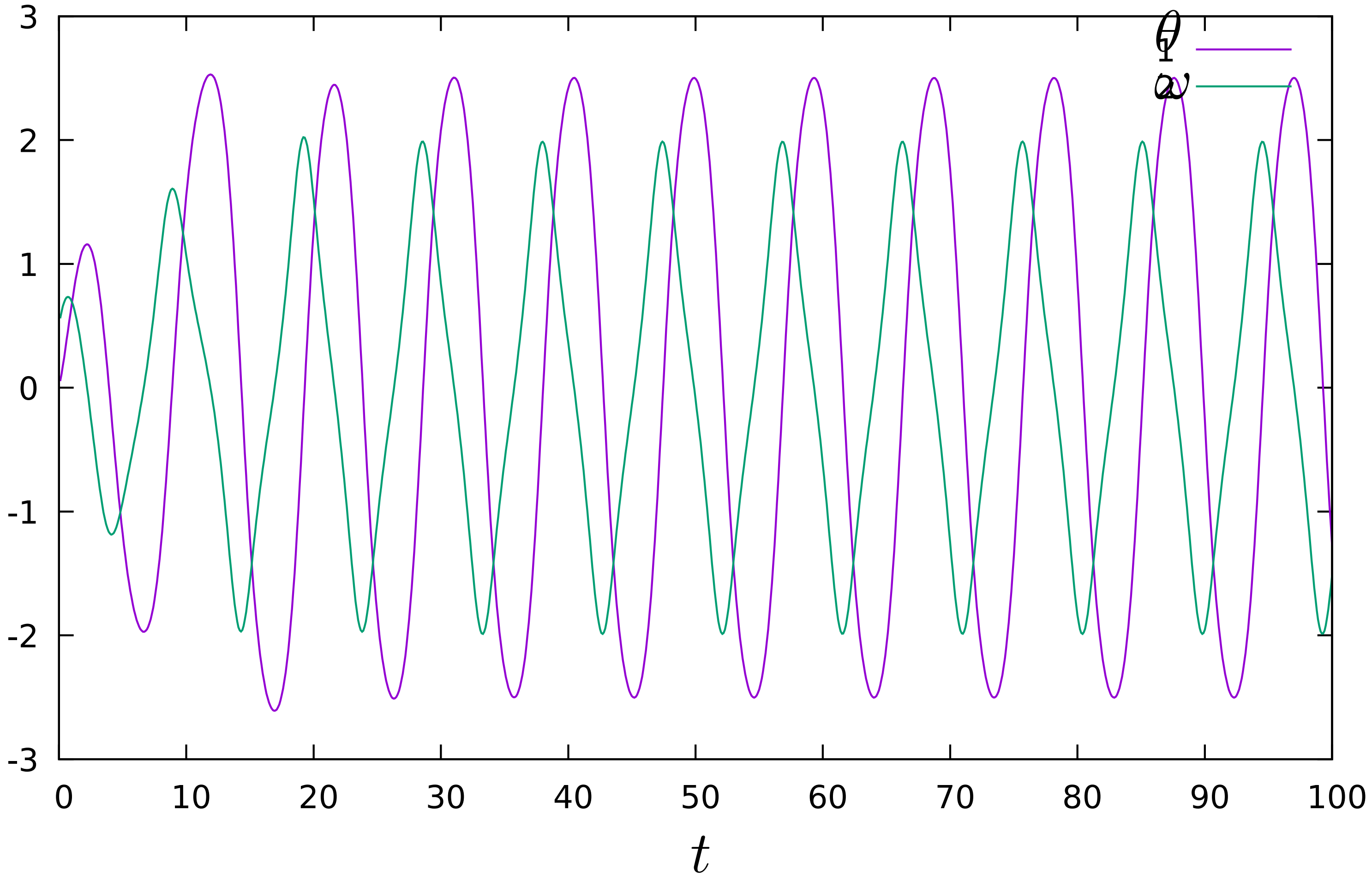
ω_0	$\frac{b}{ml^2}$	ω_d
1	1/2	2/3

$$\frac{\tau_0}{ml^2} = 0.9$$

$$\theta(t=0) = 0$$

$$\omega(t=0) = 0.5$$

$$\Delta t = 0.1$$



• Ordinary Differential Equation (Initial Value Problem)

• Driven Pendulum

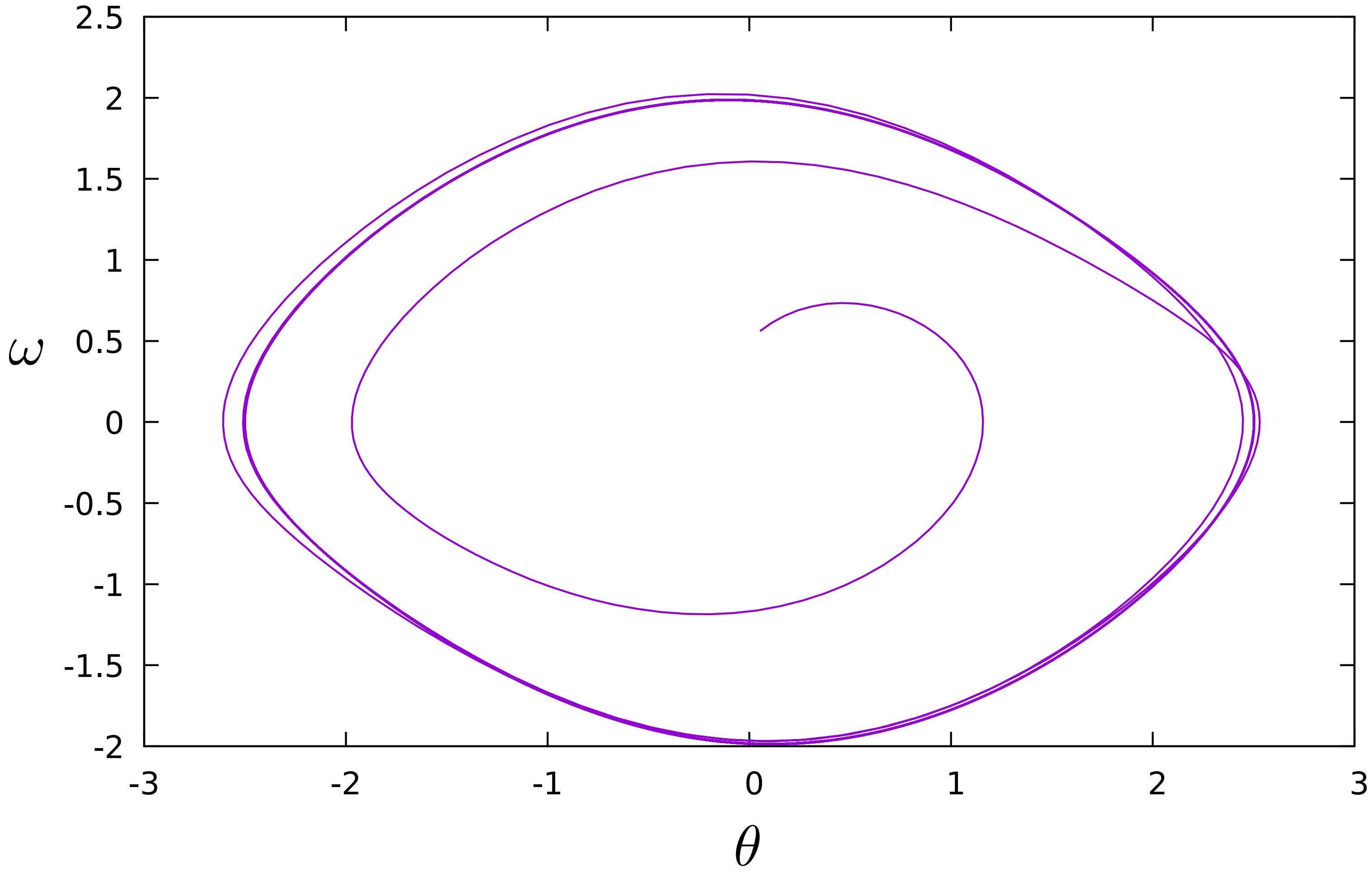
ω_0	$\dfrac{b}{ml^2}$	ω_d
1	1/2	2/3

$$\frac{\tau_0}{ml^2} = 0.9$$

$$\theta(t=0) = 0$$

$$\omega(t=0) = 0.5$$

$$\Delta t = 0.1$$



• **Ordinary Differential Equation** (Initial Value Problem)

• Driven Pendulum

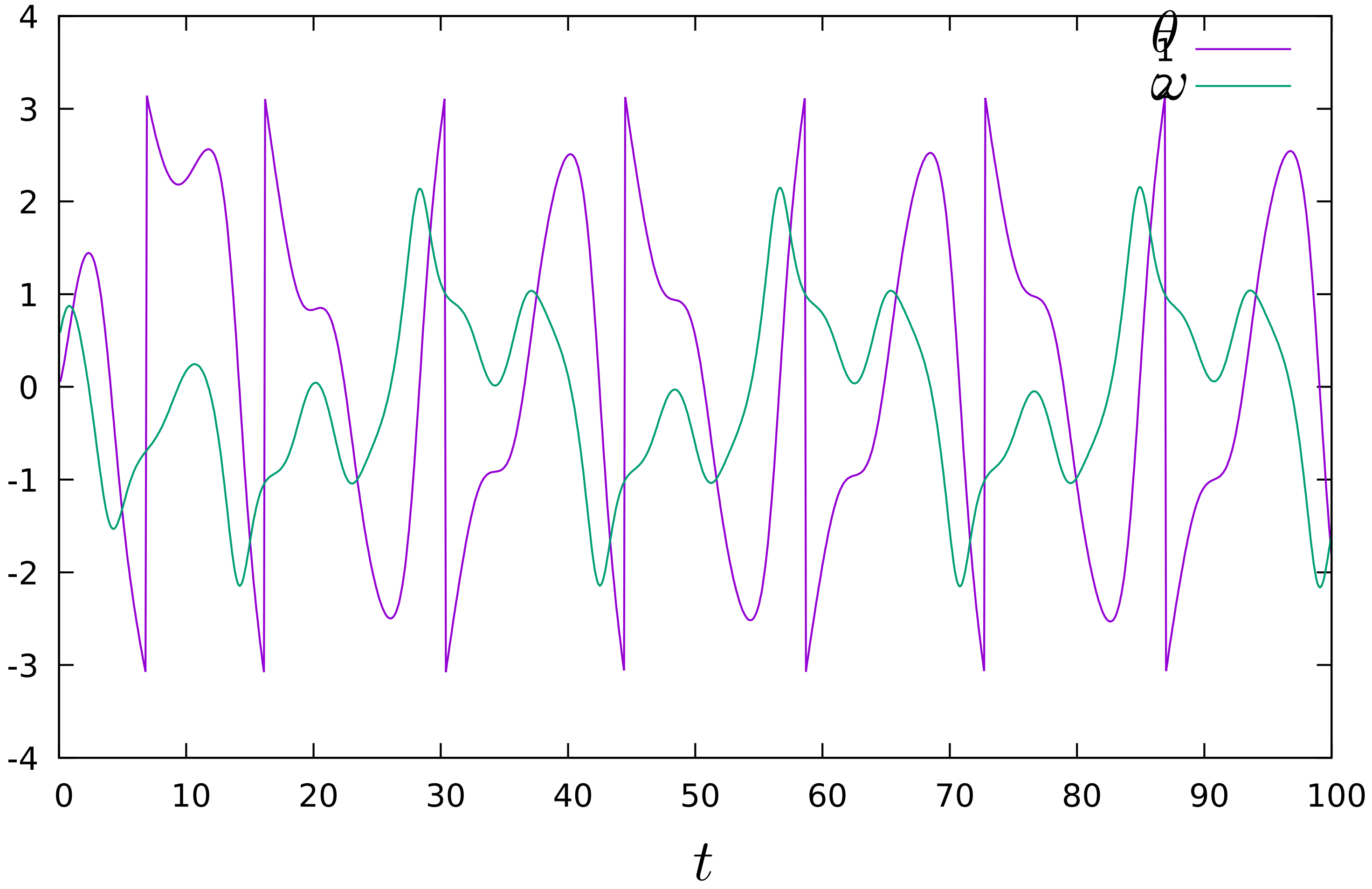
ω_0	$\frac{b}{ml^2}$	ω_{d}
1	1/2	2/3

$$\frac{\tau_0}{ml^2} = 1.15$$

$$\theta(t=0) = 0$$

$$\omega(t=0) = 0.5$$

$$\Delta t = 0.1$$



• Ordinary Differential Equation (Initial Value Problem)

• Driven Pendulum

ω_0	$\frac{b}{ml^2}$	ω_d
1	1/2	2/3

$$\frac{\tau_0}{ml^2} = 1.15$$

$$\Delta t = 0.1$$

$$\theta(t=0) = 0$$

$$\omega(t=0) = 0.5$$