Computational Physics

Lecture-25

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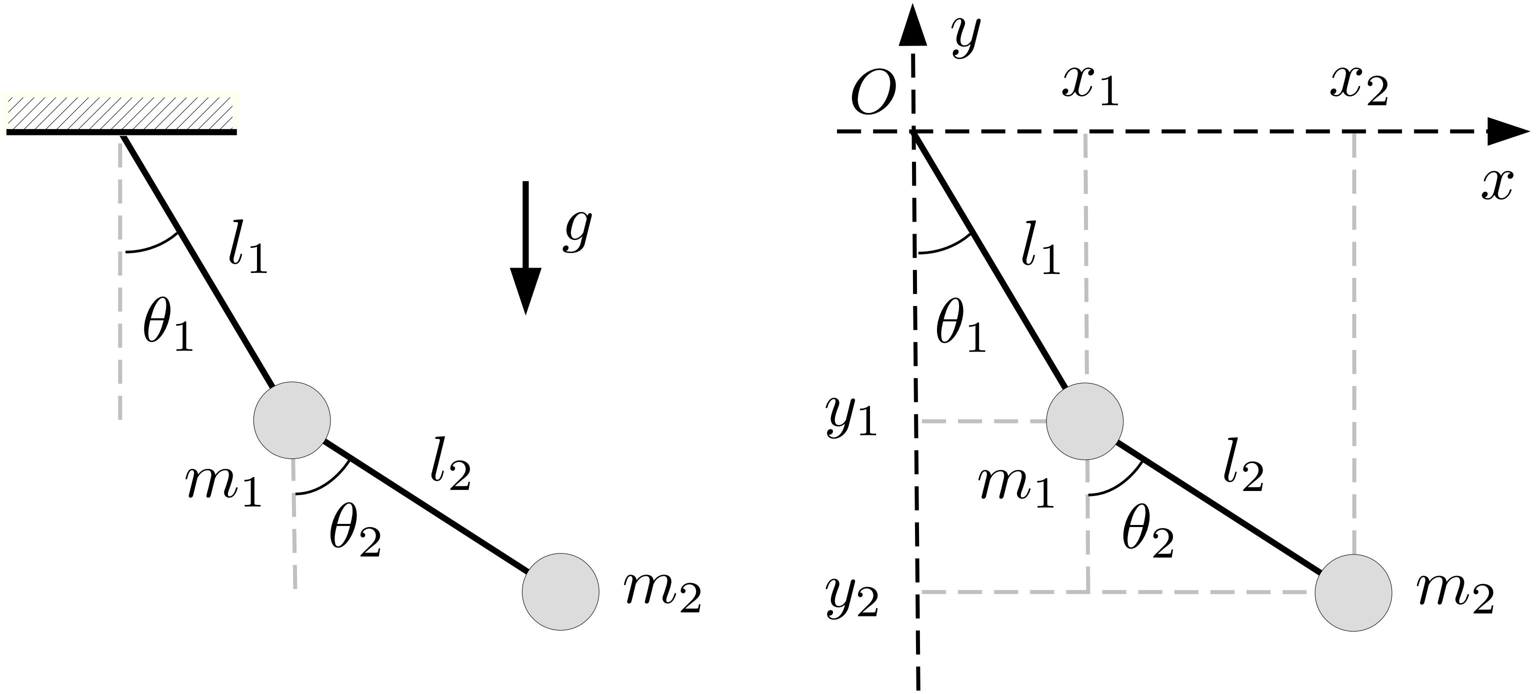
Contents

Ordinary Differential Equation

- Double Pendulum



$$
\begin{cases}
x_1 = l_1 \sin\theta_1 \\
y_1 = -l_1 \cos\theta_1 \\
x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \\
y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{cases}
$$

$$
\mathcal{L} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)
$$

Ordinary Differential Equation

- Double Pendulum

$$\begin{cases} x_1 = l_1 \sin\theta_1 \\ y_1 = -l_1 \cos\theta_1 \\ x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \end{cases}$$

$$\begin{cases} \dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1 \\ \dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1 \\ \dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 \\ \dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2 \end{cases}$$

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) \\ + m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[(m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1 - \theta_2)]$$

$$= -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g l_1\sin\theta_1$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - m_2 l_1 l_2\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2)$$

$$= -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - (m_1 + m_2)g l_1\sin\theta_1$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_2(\cancel{\dot{\theta}_1} - \dot{\theta}_2)\sin(\theta_1-\theta_2)$$

$$= -m_2 l_1 l_2\cancel{\dot{\theta}_1}\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)g l_1\sin\theta_1$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

$$= -(m_1+m_2)g l_1\sin\theta_1$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$

$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{\partial \mathcal{L}}{\partial \theta_1}$$

$$(m_1 + m_2) l_1^{\cancel{2}} \ddot{\theta}_1 + m_2 \cancel{l_1} l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 \cancel{l_1} l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$= -(m_1 + m_2) g \cancel{l_1} \sin\theta_1$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = \frac{\partial\mathcal{L}}{\partial\theta_1}$$

$$\boxed{(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) = -m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) - (m_1+m_2)g\sin\theta_1}$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1 - \theta_2)]$$

$$= m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g l_2\sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$
$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)$$
$$= m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2 g l_2\sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1 - \cancel{\dot{\theta}_2})\sin(\theta_1 - \theta_2)$$

$$= m_2 l_1 l_2\cancel{\dot{\theta}_1\dot{\theta}_2}\sin(\theta_1 - \theta_2) - m_2 g l_2\sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - m_2 l_1 l_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) = -m_2 g l_2\sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)$$

$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2}$$

$$m_2 l_2^{\cancel{2}} \ddot{\theta}_2 + m_2 l_1 \cancel{l_2} \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \cancel{l_2} \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = -m_2 g \cancel{l_2} \sin\theta_2$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = -m_2 g \sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$\cancel{m_2} l_2\ddot{\theta}_2 + \cancel{m_2} l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \cancel{m_2} l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) = -\cancel{m_2} g\sin\theta_2$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) = -g\sin\theta_2$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = \frac{\partial\mathcal{L}}{\partial\theta_2}$$

$$\boxed{l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) = l_1\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - g\sin\theta_2}$$

Ordinary Differential Equation

- Double Pendulum

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$+m_1gl_1\cos\theta_1 + m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\begin{cases} (m_1+m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) = -m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) - (m_1+m_2)g\sin\theta_1 \\ l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) + l_2\ddot{\theta}_2 = l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) - g\sin\theta_2 \end{cases}$$

$$\begin{bmatrix} (m_1+m_2)l_1 & m_2l_2\cos(\theta_1-\theta_2) \\ l_1\cos(\theta_1-\theta_2) & l_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) - (m_1+m_2)g\sin\theta_1 \\ l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) - g\sin\theta_2 \end{bmatrix}$$

Ordinary Differential Equation

- Double Pendulum

$$\begin{bmatrix} (m_1+m_2)l_1 & m_2 l_2\cos(\theta_1-\theta_2) \\ l_1\cos(\theta_1-\theta_2) & l_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1 \\ l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 \end{bmatrix}$$

$$\ddot{\theta}_1 = \frac{-(2m_1+m_2)g\sin\theta_1 - m_2 g\sin(\theta_1-2\theta_2) - 2m_2\sin(\theta_1-\theta_2)[l_2\dot{\theta}_2^2 + l_1\dot{\theta}_1^2\cos(\theta_1-\theta_2)]}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$

$$\ddot{\theta}_2 = \frac{2\sin(\theta_1-\theta_2)[(m_1+m_2)l_1 g\cos\theta_1 + (m_1+m_2)l_1\dot{\theta}_1^2 + m_2 l_2\dot{\theta}_2^2\cos(\theta_1-\theta_2)]}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$

Ordinary Differential Equation

- Double Pendulum

$$\begin{bmatrix} (m_1+m_2)l_1 & m_2 l_2 \cos(\theta_1-\theta_2) \\ l_1\cos(\theta_1-\theta_2) & l_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1 \\ l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 \end{bmatrix}$$

$$\dot{\theta}_1 = \omega_1$$

$$\dot{\theta}_2 = \omega_2$$

$$\dot{\omega}_1 = \frac{-(2m_1+m_2)g\sin\theta_1 - m_2 g \sin(\theta_1-2\theta_2) - 2m_2\sin(\theta_1-\theta_2)[l_2\omega_2^2 + l_1\omega_1^2\cos(\theta_1-\theta_2)]}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$

$$\dot{\omega}_2 = \frac{2\sin(\theta_1-\theta_2)[(m_1+m_2)l_1 g\cos\theta_1 + (m_1+m_2)l_1\omega_1^2 + m_2 l_2\omega_2^2\cos(\theta_1-\theta_2)]}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$