فیزیک ۱
دستگاه مختصات کروی

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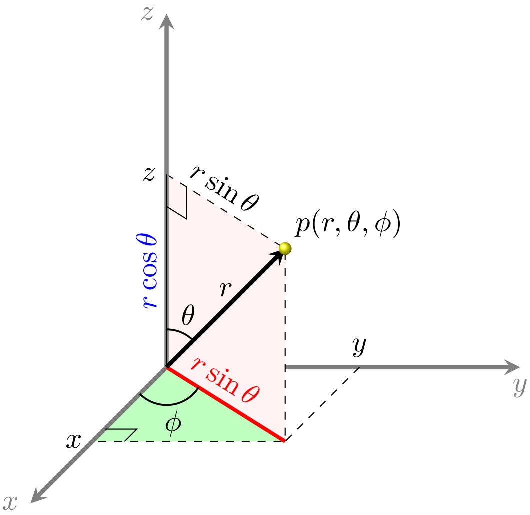
گروه فیزیک، دانشکده علوم پایه، دانشگاه قم

بهمن ۱۴۰۰

مختصه‌های دستگاه مختصات کروی



بزرگی بردار مکان

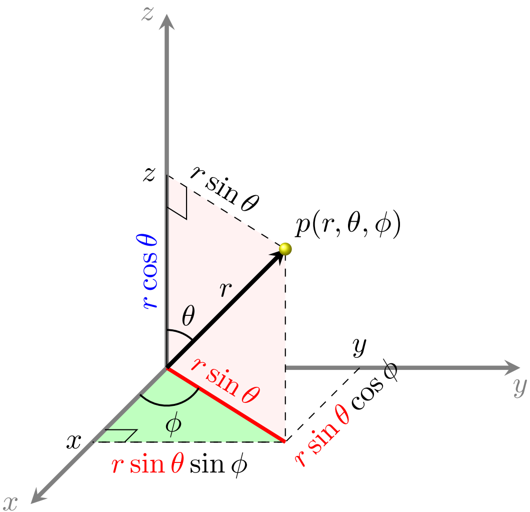
$$r = \sqrt{x^2 + y^2 + z^2}$$

جهتهای بردار مکان

$$\tan\phi = \frac{y}{x}$$

$$\cos\theta = \frac{z}{r}$$

مختصه‌های دستگاه مختصات کروی



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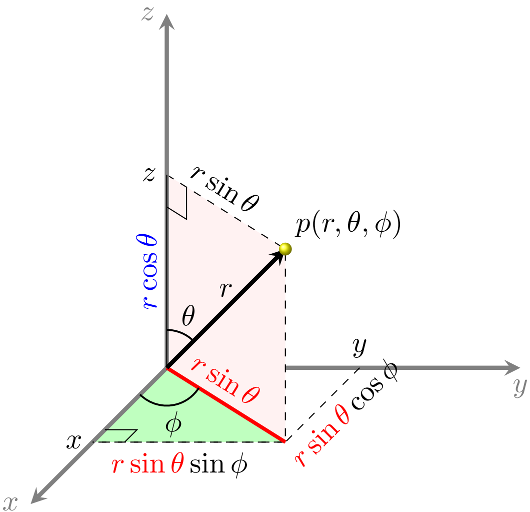
مختصه‌های دستگاه مختصات کروی



بزرگی و جهتهای بردار مکان

$$r = \sqrt{x^2 + y^2 + z^2}$$

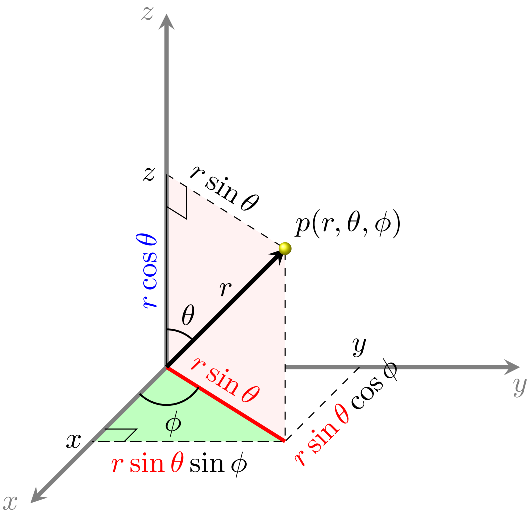
$$\tan\phi = \frac{y}{x}, \quad \cos\theta = \frac{z}{r}$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

بردار مکان

$$\vec{r} = (r\sin\theta\cos\phi)\hat{i} + (r\sin\theta\sin\phi)\hat{j} + (r\cos\theta)\hat{k}$$

مختصه‌های دستگاه مختصات کروی



بزرگی و جهتهای بردار مکان

$$r = \sqrt{x^2 + y^2 + z^2}$$

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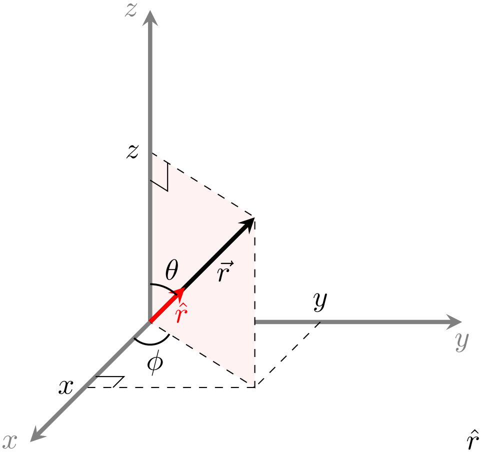
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

بردار مکان

$$\vec{r} = r[(\sin\theta\cos\phi)\hat{i} + (\sin\theta\sin\phi)\hat{j} + (\cos\theta)\hat{k}]$$

بردارهای یکه

بطور خلاصه



$$\vec{r}=\hat{i}x+\hat{j}y+\hat{k}z$$

$$(x,y,z)\rightarrow(r,\theta,\phi)$$

$$\phi=\tan^{-1}\left(\frac{y}{x}\right),\quad \theta=\cos^{-1}\left(\frac{z}{r}\right)$$

$$r=\sqrt{x^2+y^2+z^2}$$

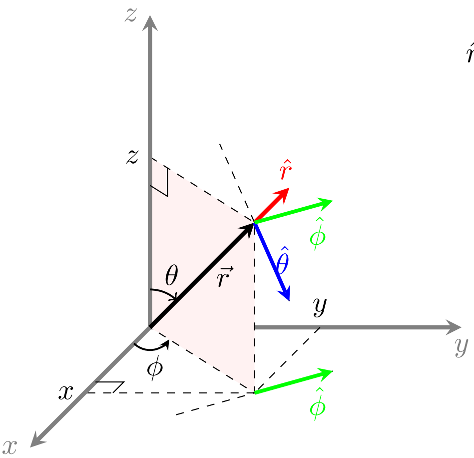
$$\begin{cases} x=r\sin\theta\cos\phi \\ y=r\sin\theta\sin\phi \\ z=r\cos\theta \end{cases}$$

$$\vec{r}=r\hat{r}$$

$$\hat{r}=\hat{i}\sin\theta\cos\phi+\hat{j}\sin\theta\sin\phi+\hat{k}\cos\theta$$

$$|\hat{r}|=1$$

بردارهای یکه



$$\vec{r} = r\hat{r}$$

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$|\hat{r}| = 1$$

از مختصات استوانه‌ای بردار یکه $\hat{\phi}$ بصورت زیر داده می‌شود،

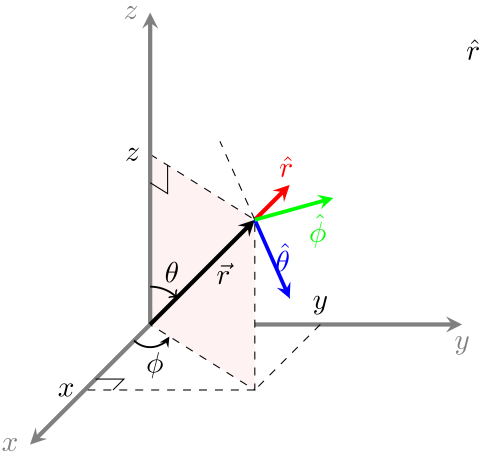
$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$|\hat{\phi}| = 1$$

بردارهای یکه $\hat{r}$ و $\hat{\theta}$ **در صفحه‌ی صورتی رنگ** قرار دارند و بر هم و بر بردار یکه $\hat{\phi}$ عمودند،

$$\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\phi} = 0$$

بردارهای یکه



$$\vec{r} = r\hat{r}$$

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\phi} = 0$$

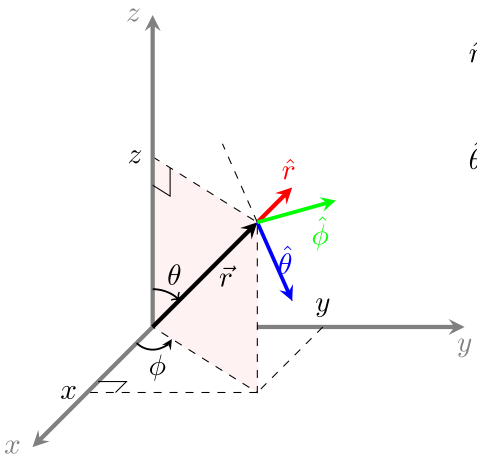
بردار یکه $\hat{\theta}$ را می‌توان از خواص راستگردی دستگاههای مختصات بدست آورد.

$$\hat{\phi}\times\hat{r} = \hat{\theta}$$

$$\hat{\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi & \cos\phi & 0 \\ sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$

بردارهای یکه



$$\vec{r} = r\hat{r}$$

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$$

بردارهای یکه‌ی تابع ϕ و θ هستند و جهت آن با تغییر ϕ و θ تغییر می‌کند.

بردارهای یکه دو به دو برهم عمودند،

$$\hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\phi} = 0$$

بردارهای یکه راستگردند،

$$\hat{\phi}\times\hat{r} = \hat{\theta}, \quad \hat{r}\times\hat{\theta} = \hat{\phi}, \quad \hat{\theta}\times\hat{\phi} = \hat{r}$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$
$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$
$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\vec{r} = r\hat{r}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{r})$$
$$\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$$
$$\hat{r} = \hat{r}(\theta, \phi)$$

$$\mathrm{d}\hat{r} = \frac{\partial\hat{r}}{\partial\theta}\mathrm{d}\theta + \frac{\partial\hat{r}}{\partial\phi}\mathrm{d}\phi$$
$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\partial\hat{r}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial\hat{r}}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi}$$
$$\frac{\partial\hat{r}}{\partial\theta} = ?, \qquad \frac{\partial\hat{r}}{\partial\phi} = ?$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\begin{cases} \dfrac{\partial\hat{r}}{\partial\theta} & = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} = \hat{\theta} \\ \dfrac{\partial\hat{r}}{\partial\phi} & = -\sin\theta\sin\phi\hat{i} + \sin\theta\cos\phi\hat{j} = \sin\theta\hat{\phi} \end{cases}$$

$$\vec{v} = \dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi}$$

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\hat{\phi}$$

$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}}$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$$

$$\begin{aligned}\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} &= \ddot{r}\hat{r} + \dot{r}\left(\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}\right)\\ &+ \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\left(\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}\right)\\ &+ \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} + r\dot{\phi}\sin\theta\left(\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t}\right)\end{aligned}$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \quad \frac{\partial\hat{r}}{\partial\phi} = \sin\theta\hat{\phi}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\phi}\sin\theta\hat{\phi}$$

$$+ \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\left(\frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi}\right)$$

$$+ \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} + r\dot{\phi}\sin\theta\left(\frac{\partial\hat{\phi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi}\right)$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \quad \frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\hat{\phi}$$

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} \\ &\quad + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + r\dot{\theta}\dot{\phi}\cos\theta\hat{\phi} \\ &\quad + \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} + r\dot{\phi}\sin\theta\left(\frac{\partial\hat{\phi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi}\right)\end{aligned}$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\frac{\partial\hat{\phi}}{\partial\theta} = 0, \quad \frac{\partial\hat{\phi}}{\partial\phi} = -\cos\phi\hat{i} - \sin\phi\hat{j} = -(\sin\theta\hat{r} + \cos\theta\hat{\theta})$$

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} \\ &+ \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + r\dot{\theta}\dot{\phi}\cos\theta\hat{\phi} \\ &+ \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} - r\dot{\theta}\dot{\phi}\sin^2\theta\hat{r} - r\dot{\phi}^2\sin\theta\cos\theta\hat{\theta}\end{aligned}$$

بردارهای سرعت و شتاب

$$r = r(t), \quad \theta = \theta(t), \quad \phi = \phi(t)$$

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

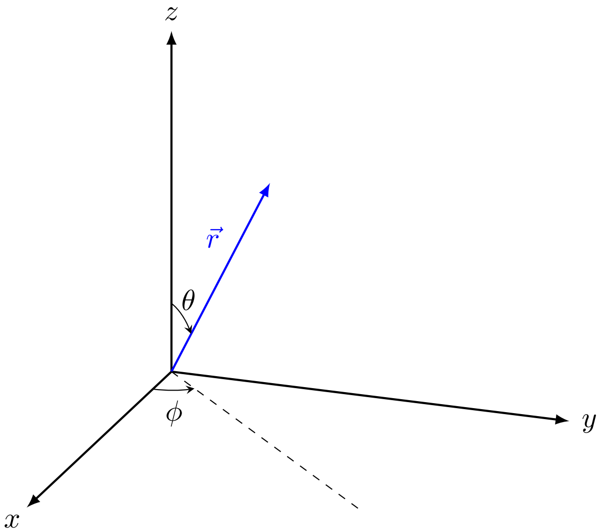
$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

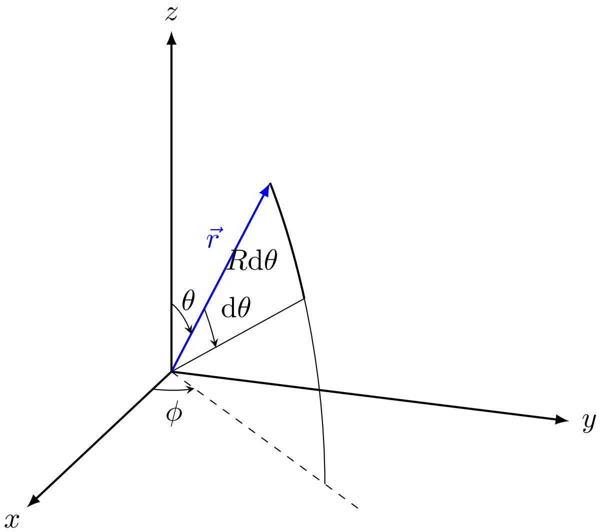
$$\begin{aligned}\vec{a} = &(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r}\\ &+ (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta}\\ &+ (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\hat{\phi}\end{aligned}$$

$$\begin{cases} a_r &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta \\ a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta \\ a_\phi &= 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta \end{cases}$$

انتگرالگیری دوبعدی-پوسته کروی



انتگرالگیری دوبعدی-پوسته کروی



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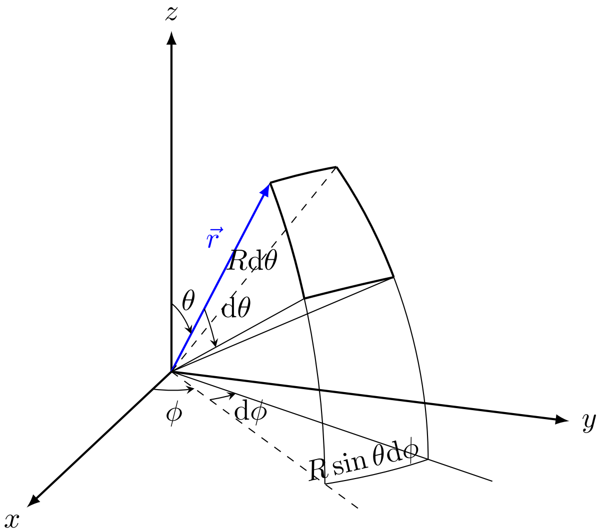
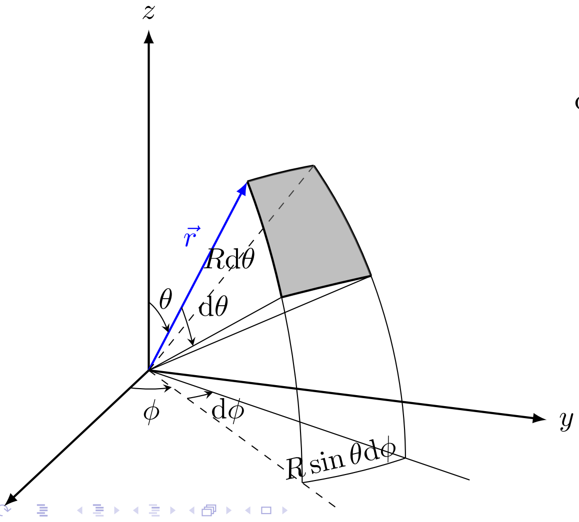


انتگرالگیری دوبعدی-پوسته کروی

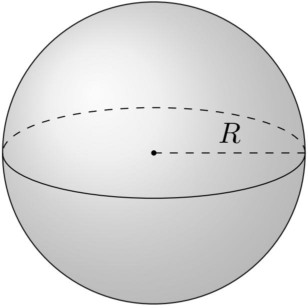


سطح تشکیل شده از دیفرانسیلهای $R\mathrm{d}\theta$ و $R\sin\theta\mathrm{d}\phi$

$$\mathrm{d}S = (R\mathrm{d}\theta)(R\sin\theta\mathrm{d}\phi)$$

$$\mathrm{d}S = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$$

انتگرالگیری دوبعدی-پوسته کروی



$$dS = R^2 \sin\theta d\theta d\phi$$

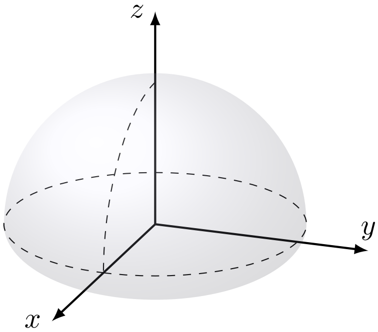
$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$

$$S = R^2 \left(\int_0^{\pi} \sin\theta d\theta \right) \left(\int_0^{2\pi} d\phi \right)$$

$$S = R^2 \left[-\cos\theta \right]_0^{\pi} \left[\phi \right]_0^{2\pi} = 4\pi R^2$$

انتگرالگیری دوبعدی-پوسته کروی



$$dS = R^2 \sin\theta d\theta d\phi$$

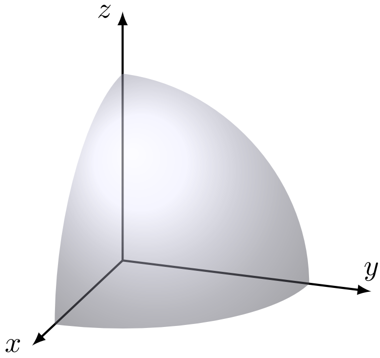
$$\boxed{0 \le \theta \le \pi/2}$$

$$0 \le \phi \le 2\pi$$

$$S = R^2 \left(\int_0^{\pi/2} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$S = R^2 \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{2\pi} = 2\pi R^2$$

انتگرالگیری دوبعدی-پوسته کروی



$$\mathrm{d}S = R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

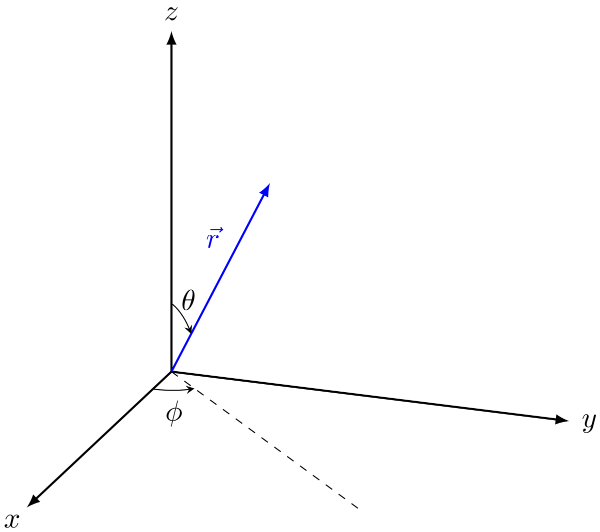
$$\boxed{0 \leq \theta \leq \pi/2}$$

$$\boxed{0 \leq \phi \leq \pi/2}$$

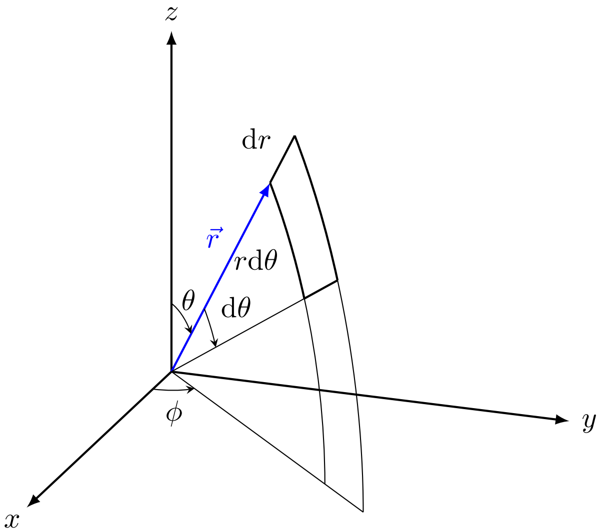
$$S = R^2 \left(\int_0^{\pi/2} \sin\theta \mathrm{d}\theta \right) \left(\int_0^{\pi/2} \mathrm{d}\phi \right)$$

$$S = R^2 \left[-\cos\theta \right]_0^{\pi/2} \left[\phi \right]_0^{\pi/2} = \frac{1}{2}\pi R^2$$

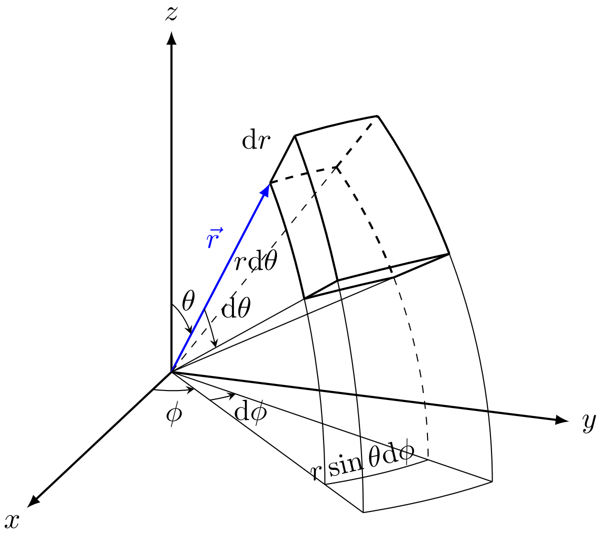
انتگرالگیری سه‌بعدی



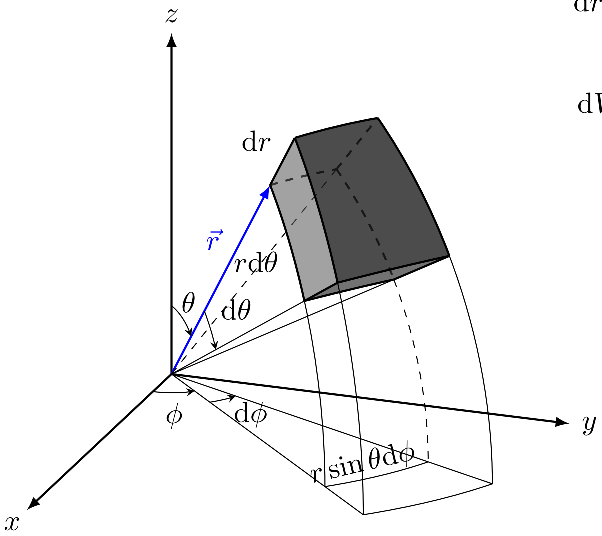
انتگرال‌گیری سه‌بعدی



انتگرالگیری سه‌بعدی



انتگرالگیری سه‌بعدی

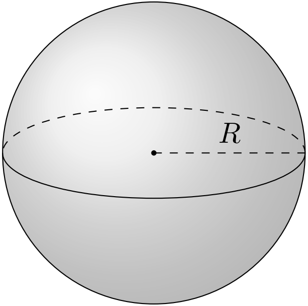


حجم تشکیل شده از دیفرانسیلهای $\mathrm{d}r$ ، $r\mathrm{d}\theta$ و $r\sin\theta\mathrm{d}\phi$

$$\mathrm{d}V = (\mathrm{d}r)(r\mathrm{d}\theta)(r\sin\theta\mathrm{d}\phi)$$

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

انتگرالگیری سه‌بعدی



$$dV = r^2 dr \sin\theta d\theta d\phi$$

$$0 \leq r \leq R$$

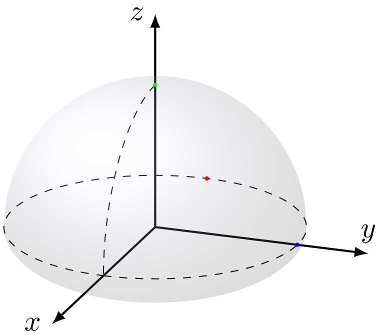
$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

$$V = \left(\int_0^R r^2 dr\right)\left(\int_0^\pi \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$V = \left[\frac{1}{3}r^3\right]_0^R [-\cos\theta]_0^\pi [\phi]_0^{2\pi} = \frac{4}{3}\pi R^3$$

انتگرالگیری سه‌بعدی



$$dV = r^2 dr \sin\theta d\theta d\phi$$

$$0 \le r \le R$$

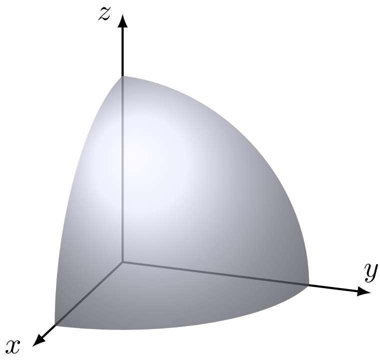
$$\boxed{0 \le \theta \le \pi/2}$$

$$0 \le \phi \le 2\pi$$

$$V = \left(\int_0^R r^2 dr\right)\left(\int_0^{\pi/2} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$

$$V = \left[\frac{1}{3}r^3\right]_0^R [-\cos\theta]_0^{\pi/2} [\phi]_0^{2\pi} = \frac{2}{3}\pi R^3$$

انتگرال‌گیری سه‌بعدی



$$\mathrm{d}V = r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$0 \le r \le R$$

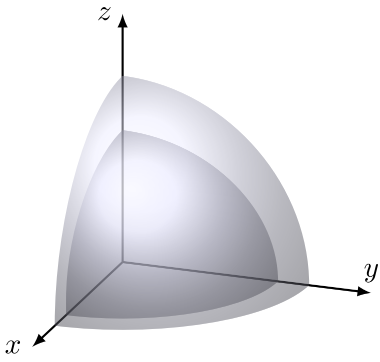
$$\boxed{0 \le \theta \le \pi/2}$$

$$\boxed{0 \le \phi \le \pi/2}$$

$$V = \left(\int_0^R r^2 \mathrm{d}r\right)\left(\int_0^{\pi/2} \sin\theta \mathrm{d}\theta\right)\left(\int_0^{\pi/2} \mathrm{d}\phi\right)$$

$$V = \left[\frac{1}{3}r^3\right]_0^R [-\cos\theta]_0^{\pi/2} [\phi]_0^{\pi/2} = \frac{1}{6}\pi R^3$$

انتگرالگیری سه‌بعدی



$$\mathrm{d}V = r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$\boxed{R_1 \le r \le R_2}$$

$$0 \le \theta \le \pi/2$$

$$0 \le \phi \le \pi/2$$

$$V = \left(\int_{R_1}^{R_2} r^2 \mathrm{d}r\right)\left(\int_0^{\pi/2} \sin\theta \mathrm{d}\theta\right)\left(\int_0^{\pi/2} \mathrm{d}\phi\right)$$

$$V = \left[\frac{1}{3} r^3\right]_{R_1}^{R_2} \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2} = \frac{1}{6}\pi(R_2^3 - R_1^3)$$