

جلسه شانزدهم

مکانیک آماری

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دانشگاه قم
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مایع کوانتومی غیر برهمکنشی

* مایع کوانتومی تحت بررسی تابع پاشندگی برابر با $E_k = \frac{\hbar^2 k^2}{2m}$ و اسپینی برابر با S دارد.

* تابع پارش بزرگ برای فرمیونها و بوزونها بوسیله رابطه زیر داده می شود،

$$\ln \mathcal{Z}_{\pm} = \pm \ln[1 \pm e^{-\beta(E_k - \mu)}]$$

فرمیونها
بوزون

* پتانسیل کانونیک بزرگ برای تابع پارش داده شده بصورت زیر بدست می آید

$$\Phi_G = -(2S + 1)k_B T \ln \mathcal{Z} = \mp (2S + 1)k_B T \sum_k \ln[1 \pm e^{-\beta(E_k - \mu)}]$$

که $2S + 1$ تعداد حالت های اسپینی می باشد.

مایع کوانتومی غیر برهمکنشی

$$\Phi_G = -(2S + 1)k_B T \ln \mathcal{Z} = \mp (2S + 1)k_B T \sum_k \ln[1 \pm e^{-\beta(E_k - \mu)}]$$

$$(2S + 1) \sum_k \rightarrow (2S + 1) \frac{L^d}{(2\pi)^d} \int d^d k \rightarrow \int dE g_d(E)$$

$$\Phi_G = \mp k_B T \int dE g_d(E) \ln[1 \pm e^{-\beta(E - \mu)}]$$

چگالی حالت‌ها

$$g(E) = (2S + 1) \sum_k \delta\left(\frac{\hbar^2 k^2}{2m} - E\right), \quad \begin{cases} g_{3D}(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E} \\ g_{2D}(E) = (2S + 1) \frac{A}{2\pi} \frac{m}{\hbar^2} \\ g_{1D}(E) = (2S + 1) \frac{L}{2\pi} \frac{1}{2} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{\sqrt{E}} \end{cases}$$

مايع كوانتومي غير برهمکنشي

$$\begin{array}{l} \vdots \\ E_3 \quad \frac{n_3}{\langle n_3 \rangle} \\ E_2 \quad \frac{n_2}{\langle n_2 \rangle} \\ E_1 \quad \frac{n_1}{\langle n_1 \rangle} \end{array} \quad \langle n_i \rangle_{\text{FD}} = \frac{1}{e^{\beta(E_i - \mu)} + 1} = f_{\text{FD}}(E_i)$$
$$\langle n_i \rangle_{\text{BE}} = \frac{1}{e^{\beta(E_i - \mu)} - 1} = f_{\text{BE}}(E_i)$$

فرد ميونيوني

$$N = (2S + 1) \sum_k \langle n_k \rangle_{\text{FD}}$$
$$U = (2S + 1) \sum_k E_k \langle n_k \rangle_{\text{FD}}$$

بوزونيوني

$$N = (2S + 1) \sum_k \langle n_k \rangle_{\text{BE}}$$
$$U = (2S + 1) \sum_k E_k \langle n_k \rangle_{\text{BE}}$$

مايع كوانتومي غير برهمکنشي

$$N = (2S + 1) \sum_k \langle n_k \rangle_{\text{FD}}$$

$$U = (2S + 1) \sum_k E_k \langle n_k \rangle_{\text{FD}}$$

$$N = \int_0^{\infty} dE g_d(E) f_{\text{FD}}(E)$$

$$U = \int_0^{\infty} dE E g_d(E) f_{\text{FD}}(E)$$

$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

$$N = (2S + 1) \sum_k \langle n_k \rangle_{\text{BE}}$$

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$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

فرميونيها

بوزونيها

مایع کوانتومی غیر برهمکنشی

فرمیونها

$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

بوزونها

$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

$$g_{3D}(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E}, \quad z = e^{\beta\mu}$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{1/2} dE}{e^{\beta(E-\mu)} + 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{1/2} dE}{z^{-1} e^{\beta E} + 1}$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{3/2} dE}{e^{\beta(E-\mu)} + 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{3/2} dE}{z^{-1} e^{\beta E} + 1}$$

مایع کوانتومی غیر برهمکنشی

فرمیونها

$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} + 1}$$

بوزونها

$$N = \int_0^{\infty} \frac{g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

$$U = \int_0^{\infty} \frac{E g_d(E) dE}{e^{\beta(E-\mu)} - 1}$$

$$g_{3D}(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E}, \quad z = e^{\beta\mu}$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{1/2} dE}{e^{\beta(E-\mu)} - 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{1/2} dE}{z^{-1} e^{\beta E} - 1}$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{3/2} dE}{e^{\beta(E-\mu)} - 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\infty} \frac{E^{3/2} dE}{z^{-1} e^{\beta E} - 1}$$

مایع کوانتومی غیر برهمکنشی

تابع گاما

↑

اتحاد مفید $\int_0^{\infty} \frac{E^{n-1} dE}{z^{-1} e^{\beta E} \pm 1} = (k_B T)^n \Gamma(n) [\mp \text{Li}_n(\mp z)], \quad z = e^{\beta \mu}$

↓

تابع پلی لگاریتم

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (k_B T)^{3/2} \Gamma(3/2) [\mp \text{Li}_{3/2}(\mp z)], \quad \Gamma(3/2) = \sqrt{\pi}/2$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{3/2} (\sqrt{\pi}/2) [\mp \text{Li}_{3/2}(\mp z)] = (2S + 1) \frac{V}{\lambda_{\text{th}}^3} [\mp \text{Li}_{3/2}(\mp z)]$$

$$N = (2S + 1) \frac{V}{\lambda_{\text{th}}^3} [\mp \text{Li}_{3/2}(\mp z)], \quad \lambda_{\text{th}} = \frac{h}{\sqrt{2m\pi k_B T}}$$

مایع کوانتومی غیر برهمکنشی

تابع گاما

↑

اتحاد مفید $\int_0^\infty \frac{E^{n-1} dE}{z^{-1} e^{\beta E} \pm 1} = (k_B T)^n \Gamma(n) [\mp \text{Li}_n(\mp z)], \quad z = e^{\beta \mu}$

↓

تابع پلی لگاریتم

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (k_B T)^{5/2} \Gamma(5/2) [\mp \text{Li}_{5/2}(\mp z)], \quad \Gamma(5/2) = 3\sqrt{\pi}/4$$

$$U = (2S + 1) \frac{3}{2} k_B T \frac{V}{(2\pi)^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{3/2} (\sqrt{\pi}/2) [\mp \text{Li}_{5/2}(\mp z)] = (2S + 1) \frac{3}{2} k_B T \frac{V}{\lambda_{\text{th}}^3} [\mp \text{Li}_{5/2}(\mp z)]$$

$$U = (2S + 1) \frac{3}{2} k_B T \frac{V}{\lambda_{\text{th}}^3} [\mp \text{Li}_{5/2}(\mp z)], \quad \lambda_{\text{th}} = \frac{h}{\sqrt{2m\pi k_B T}}$$

مایع کوانتومی غیر برهمکنشی

$$N = (2S + 1) \frac{V}{\lambda_{\text{th}}^3} [\text{Li}_{3/2}(\mp z)]$$

$$\frac{N}{[\text{Li}_{3/2}(\mp z)]} = (2S + 1) \frac{V}{\lambda_{\text{th}}^3}$$

$$U = (2S + 1) \frac{3}{2} k_B T \frac{V}{\lambda_{\text{th}}^3} [\text{Li}_{5/2}(\mp z)] = \frac{3}{2} N k_B T \frac{[\text{Li}_{5/2}(\mp z)]}{[\text{Li}_{3/2}(\mp z)]}$$

$$U = \frac{3}{2} N k_B T \frac{\text{Li}_{5/2}(\mp z)}{\text{Li}_{3/2}(\mp z)}$$

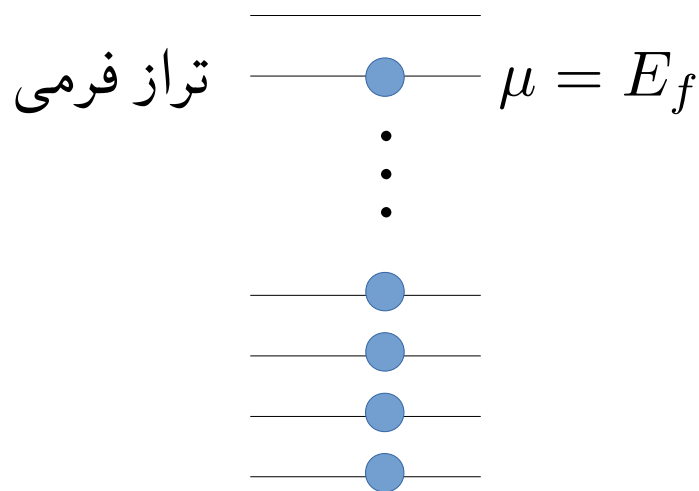
حالت خاص

$$z = e^{\beta\mu} \ll 1 \Rightarrow \text{Li}_n(z) \approx z$$

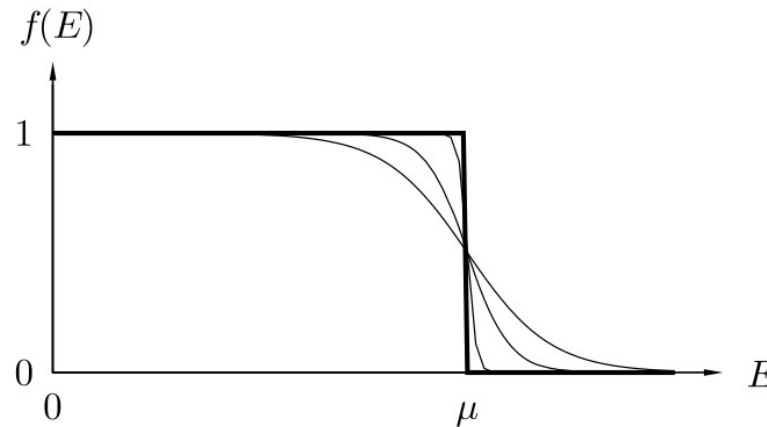
$$N \approx (2S + 1) \frac{V}{\lambda_{\text{th}}^3} z, \quad U \approx \frac{3}{2} N k_B T$$

مایع کوانتومی غیر برهمکنشی

$$T = 0$$



$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} = \begin{cases} 1, & E \leq \mu \\ 0, & E > \mu \end{cases}$$

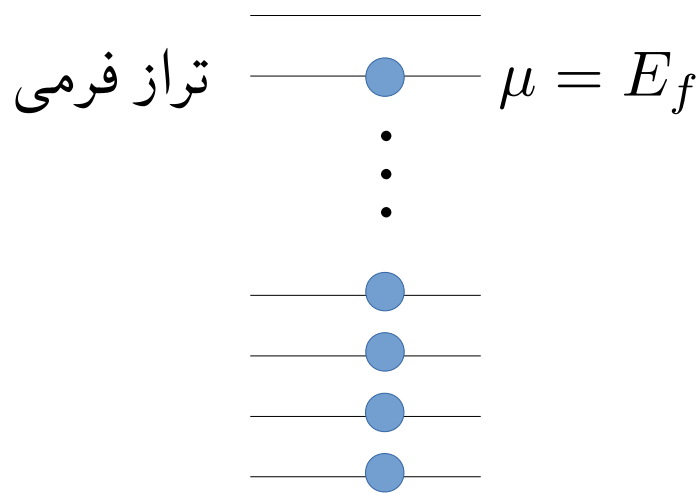


$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{E^{1/2} dE}{e^{\beta(E-\mu)} + 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{E_f} E^{1/2} dE$$

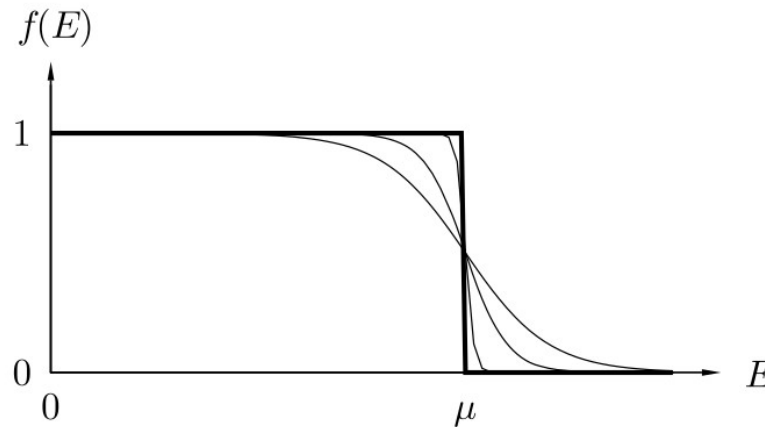
$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (2/3) E_f^{3/2}$$

مایع کوانتومی غیر برهمکنشی

$$T = 0$$



$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} = \begin{cases} 1, & E \leq \mu \\ 0, & E > \mu \end{cases}$$

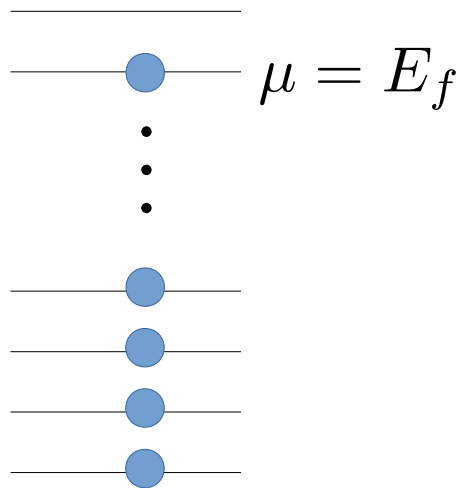


$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{E^{3/2} dE}{e^{\beta(E-\mu)} + 1} = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{E_f} E^{3/2} dE$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (2/5) E_f^{5/2}$$

مایع کوانتومی غیر برهمکنشی

$$T = 0$$



$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (2/3) E_f^{3/2}$$

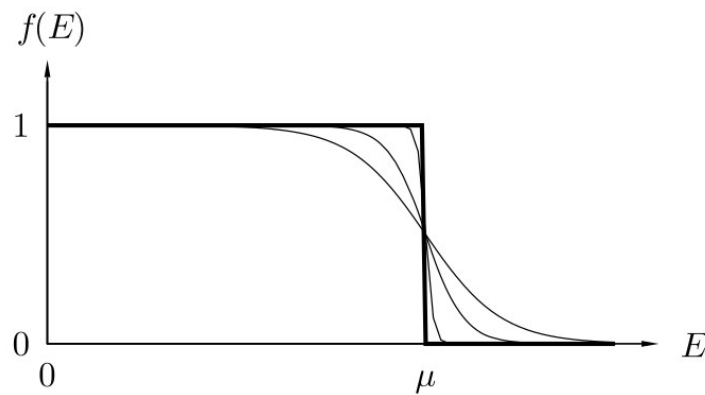
$$n = \frac{N}{V} = (2S + 1) \frac{1}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (2/3) E_f^{3/2}$$

$$E_f = \frac{\hbar^2}{2m} \left[\frac{6\pi^2 n}{2S + 1} \right]^{2/3}, \quad n = \frac{N}{V}$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} (2/5) E_f^{5/2}$$

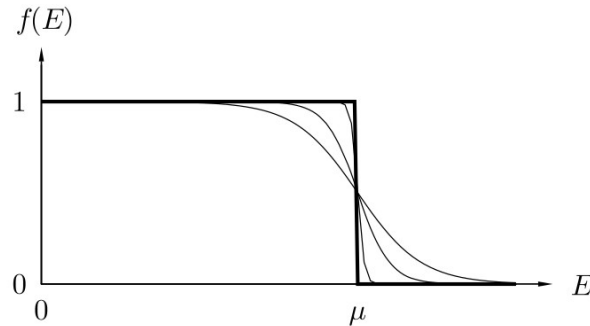
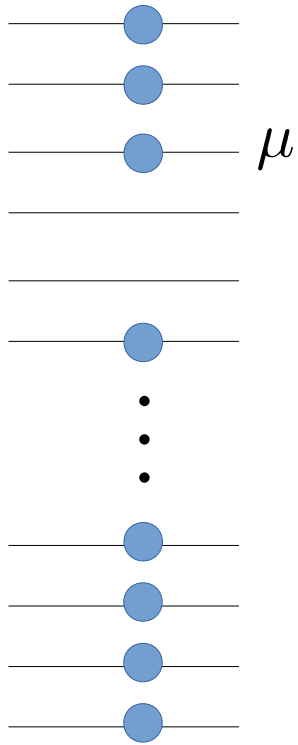
$$\frac{U}{N} = \frac{2/5 E_f^{5/2}}{2/3 E_f^{3/2}} = \frac{3}{5} E_f$$

$$U = \frac{3}{5} N E_f$$



مایع کوانتومی غیر برهمکنشی

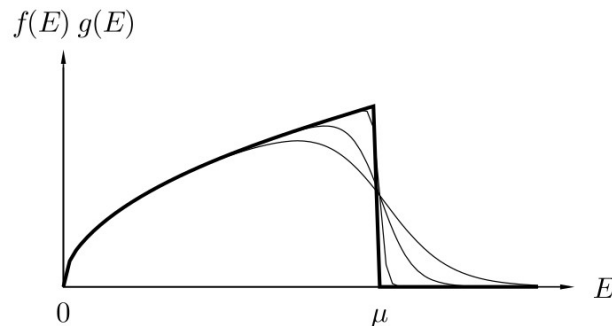
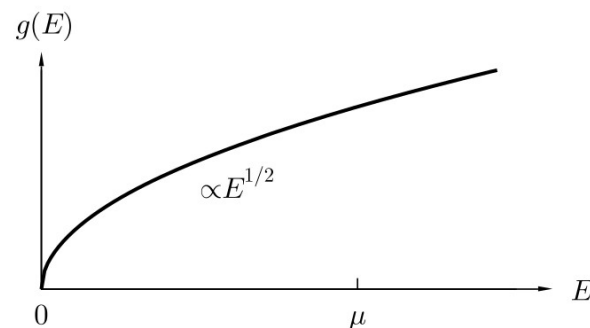
$T \neq 0$



$$N = \int_0^{\infty} dE g(E) f(E)$$

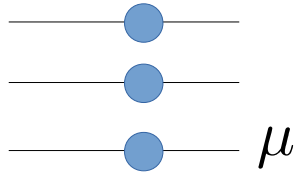
$$U = \int_0^{\infty} dE E g(E) f(E)$$

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1}$$



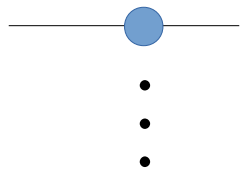
$$g(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E}$$

مایع کوانتومی غیر برهمکنشی

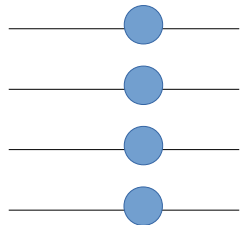


$$I = \int_0^{\infty} \frac{\phi(E)}{e^{\beta(E-\mu)} + 1} dE \quad T \neq 0$$

$$\beta(E - \mu) = x \Rightarrow E = k_B T x + \mu \Rightarrow dE = k_B T dx$$



$$I = k_B T \int_{-\mu/k_B T}^{\infty} \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

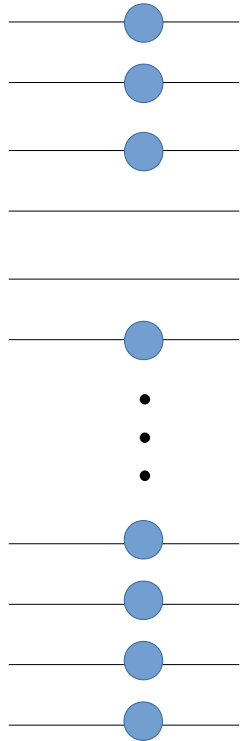


$$I = k_B T \int_{-\mu/k_B T}^0 \frac{\phi(k_B T x + \mu)}{e^x + 1} dx + k_B T \int_0^{\infty} \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

$$\int_{-\mu/k_B T}^0 \frac{\phi(k_B T x + \mu)}{e^x + 1} dx \quad \boxed{x \rightarrow -x \Rightarrow dx \rightarrow -dx} \quad \int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^{-x} + 1} dx$$

$$\boxed{\frac{1}{e^{-x} + 1} = 1 - \frac{1}{e^x + 1}} \quad \int_0^{\mu/k_B T} \phi(-k_B T x + \mu) dx - \int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx$$

مایع کوانتومی غیر برهمکنشی



μ

$$I = k_B T \int_0^{\mu/k_B T} \phi(-k_B T x + \mu) dx$$

$T \neq 0$

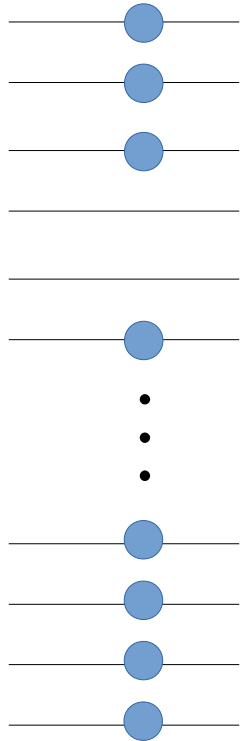
$$-k_B T \int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx + k_B T \int_0^{\infty} \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

$$z = -k_B T x + \mu \rightarrow dz = -k_B T dx$$

$$k_B T \int_0^{\mu/k_B T} \phi(-k_B T x + \mu) dx = \int_0^{\mu} \phi(z) dz$$

$$I = \int_0^{\mu} \phi(z) dz - k_B T \int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx + k_B T \int_0^{\infty} \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

مایع کوانتومی غیر برهمکنشی



μ

$$I = \int_0^\mu \phi(z) dz$$

$T \neq 0$

$$-k_B T \int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx + k_B T \int_0^\infty \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

$$\boxed{\frac{\mu}{k_B T} \gg 1} \quad \text{تقریب}$$

$$\int_0^{\mu/k_B T} \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx \approx \int_0^\infty \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx$$

$$I = \int_0^\mu \phi(z) dz - k_B T \int_0^\infty \frac{\phi(-k_B T x + \mu)}{e^x + 1} dx + k_B T \int_0^\infty \frac{\phi(k_B T x + \mu)}{e^x + 1} dx$$

$$I = \int_0^\mu \phi(z) dz + k_B T \int_0^\infty \frac{1}{e^x + 1} [\phi(k_B T x + \mu) - \phi(-k_B T x + \mu)] dx$$

مایع کوانتومی غیر برهمکنشی

$$\boxed{\frac{\mu}{k_B T} \gg 1} \quad \text{تقریب} \quad T \neq 0$$

$$I = \int_0^\mu \phi(z) dz + k_B T \int_0^\infty \frac{1}{e^x + 1} [\phi(\mu + k_B T x) - \phi(\mu - k_B T x)] dx$$

$$\begin{cases} \phi(\mu + k_B T x) = \phi(\mu) + k_B T x \phi^{(1)}(\mu) + \frac{1}{2!} (k_B T)^2 x^2 \phi^{(2)}(\mu) + \frac{1}{3!} (k_B T)^3 x^3 \phi^{(3)}(\mu) + \dots \\ \phi(\mu - k_B T x) = \phi(\mu) - k_B T x \phi^{(1)}(\mu) + \frac{1}{2!} (k_B T)^2 x^2 \phi^{(2)}(\mu) - \frac{1}{3!} (k_B T)^3 x^3 \phi^{(3)}(\mu) + \dots \end{cases}$$

$$\phi(\mu + k_B T x) - \phi(\mu - k_B T x) = 2k_B T x \phi^{(1)}(\mu) + \frac{1}{3} (k_B T)^3 x^3 \phi^{(3)}(\mu) + \dots$$

$$I = \int_0^\mu \phi(z) dz + 2(k_B T)^2 \phi^{(1)}(\mu) \int_0^\infty \frac{x}{e^x + 1} dx + \frac{1}{3} (k_B T)^4 \phi^{(3)}(\mu) \int_0^\infty \frac{x^3}{e^x + 1} dx + \dots$$

مایع کوانتومی غیر برهمکنشی

$$\boxed{\frac{\mu}{k_B T} \gg 1} \quad \text{تقریب} \quad T \neq 0$$

$$I = \int_0^\mu \phi(z) dz + 2(k_B T)^2 \phi^{(1)}(\mu) \int_0^\infty \frac{x}{e^x + 1} dx + \frac{1}{3}(k_B T)^4 \phi^{(3)}(\mu) \int_0^\infty \frac{x^3}{e^x + 1} dx + \dots$$

$$\int_0^\infty \frac{x}{e^x + 1} dx = \frac{\pi^2}{12}, \quad \int_0^\infty \frac{x^3}{e^x + 1} dx = \frac{7\pi^4}{120}$$

$$I = \int_0^\mu \phi(z) dz + \frac{\pi^2}{6} (k_B T)^2 \phi^{(1)}(\mu) + \frac{7\pi^4}{360} (k_B T)^4 \phi^{(3)}(\mu) + \dots$$

$$\phi(E) = g(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E}$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \left[\frac{2}{3} \mu^{3/2} + \frac{\pi^2}{6} (k_B T)^2 \frac{1}{2} \mu^{-1/2} + \dots \right]$$

مایع کوانتومی غیر برهمکنشی

$$\boxed{\frac{\mu}{k_B T} \gg 1} \quad \text{تقریب}$$

$$T \neq 0$$

$$I = \int_0^\mu \phi(z) dz + \frac{\pi^2}{6} (k_B T)^2 \phi^{(1)}(\mu) + \frac{7\pi^4}{360} (k_B T)^4 \phi^{(3)}(\mu) + \dots$$

$$\phi(E) = g(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{E}$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \left[\frac{2}{3} \mu^{3/2} + \frac{\pi^2}{6} (k_B T)^2 \frac{1}{2} \mu^{-1/2} + \dots \right]$$

$$N = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \frac{2}{3} \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{k_B T}{\mu} \right)^2 + \dots \right]$$

مایع کوانتومی غیر برهمکنشی

$$\boxed{\frac{\mu}{k_B T} \gg 1} \quad \text{تقریب}$$

$$T \neq 0$$

$$I = \int_0^\mu \phi(z) dz + \frac{\pi^2}{6} (k_B T)^2 \phi^{(1)}(\mu) + \frac{7\pi^4}{360} (k_B T)^4 \phi^{(3)}(\mu) + \dots$$

$$\phi(E) = E g(E) = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} E^{3/2}$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \left[\frac{2}{5} \mu^{5/2} + \frac{\pi^2}{6} (k_B T)^2 \frac{3}{2} \mu^{1/2} + \dots \right]$$

$$U = (2S + 1) \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \frac{2}{5} \mu^{5/2} \left[1 + \frac{5\pi^2}{8} \left(\frac{k_B T}{\mu} \right)^2 + \dots \right]$$