مکانیک آماری
جلسه بیست و ششم

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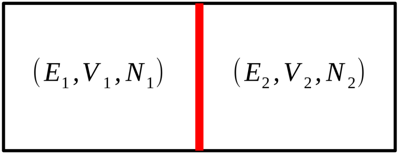
آنسامبل میکروکانونیک

نشان دهید که برای دو سیستم بزرگ در تعادل گرمایی میکروحالت نهایی تابع گوسی از اختلاف متغیر انرژی یک از سیستمها نسبت به حالت تعادلی خود است.

میکروحالت مرکب



$$\Omega(E_1, E_2) = \Omega_1(E_1)\Omega_2(E_2)$$

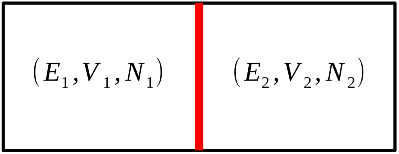
$$E = E_1 + E_2 = \text{const.}$$

قصد داریم رابطه

$$\frac{\Omega(\bar{E}_1, \bar{E}_2)}{\Omega_1(\bar{E}_1)\Omega_2(\bar{E}_2)} \simeq e^{-\alpha \delta E_1^2}$$

را بررسی کنیم که $E_1 = \bar{E}_1 + \delta E_1$.

میکروحالت مرکب



$$\Omega(E_1, E_2) = \Omega_1(E_1)\Omega_2(E_2)$$

$$E = E_1 + E_2 = \text{const.}$$

در ادامه $E_2 = E - E_1$ و از طرفین لگاریتم می گیریم

$$\ln \Omega(E, E_1) = \ln \Omega_1(E_1) + \ln \Omega_2(E - E_1)$$

دو طرف تساوی بالا بصورت تابعی از E_1 است. اگر تغییرات E_1 را حول $\bar{E}_1$ بصورت

$$E_1 \to \bar{E}_1 + \delta E_1$$

داشته باشیم،

$$\ln \Omega(E, \bar{E}_1 + \delta E_1) = \ln \Omega_1(\bar{E}_1 + \delta E_1) + \ln \Omega_2(E - \bar{E}_1 - \delta E_1)$$

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$$\ln\Omega(E,\bar{E}_1+\delta E_1)=\ln\Omega_1(\bar{E}_1+\delta E_1)+\ln\Omega_2(E-\bar{E}_1-\delta E_1)$$

اگر فرض کنیم که δE_1 کوچک باشد می توان طرف راست عبارت بالا را حول $\bar{E}_1$ بسط داد،

$$\begin{aligned}\ln\Omega(E,\bar{E}_1+\delta E_1)=&\left[\ln\Omega_1(\bar{E}_1)+\left(\frac{\partial\ln\Omega_1}{\partial\bar{E}_1}\right)\delta E_1\right.\\&\left.+\frac{1}{2}\left(\frac{\partial^2\ln\Omega_1}{\partial\bar{E}_1^2}\right)\delta E_1^2+\cdots\right]\\&+\left[\ln\Omega_2(E-\bar{E}_1)+\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_1}\right)\delta E_1\right.\\&\left.+\frac{1}{2}\left(\frac{\partial^2\ln\Omega_2}{\partial\bar{E}_1^2}\right)\delta E_1^2+\cdots\right]\end{aligned}$$

$$\bar{E}_2=E-\bar{E}_1:\ \left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_1}\right)=\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_2}\right)\left(\frac{\partial\bar{E}_2}{\partial\bar{E}_1}\right)=-\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_2}\right)$$

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$$\ln\Omega(E,\bar{E}_1+\delta E_1)=\left[\ln\Omega_1(\bar{E}_1)+\left(\frac{\partial\ln\Omega_1}{\partial\bar{E}_1}\right)\delta E_1\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial^2\ln\Omega_1}{\partial\bar{E}_1^2}\right)\delta E_1^2+\cdots\right]$$
$$+\left[\ln\Omega_2(E-\bar{E}_1)+\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_1}\right)\delta E_1\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial^2\ln\Omega_2}{\partial\bar{E}_1^2}\right)\delta E_1^2+\cdots\right]$$

$$\bar{E}_2=E-\bar{E}_1:\ \left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_1}\right)=\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_2}\right)\left(\frac{\partial\bar{E}_2}{\partial\bar{E}_1}\right)=-\left(\frac{\partial\ln\Omega_2}{\partial\bar{E}_2}\right)$$

$$\left(\frac{\partial^2\ln\Omega_2}{\partial\bar{E}_1^2}\right)=\left(\frac{\partial^2\ln\Omega_2}{\partial\bar{E}_2^2}\right)$$

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$$\ln\Omega(E,\bar{E}_1+\delta E_1) = \left[\ln\Omega_1(\bar{E}_1) + \left(\frac{\partial \ln\Omega_1}{\partial \bar{E}_1}\right)\delta E_1 + \frac{1}{2}\left(\frac{\partial^2 \ln\Omega_1}{\partial \bar{E}_1^2}\right)\delta E_1^2 + \cdots\right] + \left[\ln\Omega_2(E-\bar{E}_1) - \left(\frac{\partial \ln\Omega_2}{\partial \bar{E}_2}\right)\delta E_1 + \frac{1}{2}\left(\frac{\partial^2 \ln\Omega_2}{\partial \bar{E}_2^2}\right)\delta E_1^2 + \cdots\right]$$

$$\left(\frac{\partial \ln\Omega_1}{\partial \bar{E}_1}\right) = \frac{1}{k_B T_1}, \quad \left(\frac{\partial^2 \ln\Omega_1}{\partial \bar{E}_1^2}\right) = -\frac{1}{k_B T_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right)$$

$$\left(\frac{\partial \ln\Omega_2}{\partial \bar{E}_2}\right) = \frac{1}{k_B T_2}, \quad \left(\frac{\partial^2 \ln\Omega_2}{\partial \bar{E}_2^2}\right) = -\frac{1}{k_B T_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)$$

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$$
\begin{aligned}
\ln\Omega(E,\bar{E}_1+\delta E_1) = &\left[\ln\Omega_1(\bar{E}_1) + \frac{1}{k_BT_1}\delta E_1 \right.\\
&\left. -\frac{1}{2}\frac{1}{k_BT_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)\delta E_1^2 + \cdots\right]\\
&+\left[\ln\Omega_2(E-\bar{E}_1) - \frac{1}{k_BT_2}\delta E_1 \right.\\
&\left. -\frac{1}{2}\frac{1}{k_BT_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)\delta E_1^2 + \cdots\right]
\end{aligned}
$$

$$
\begin{aligned}
\ln\Omega(E,\bar{E}_1+\delta E_1) = \ln\Omega_1(\bar{E}_1) + \ln\Omega_2(E-\bar{E}_1)&\\
+\left[\frac{1}{k_BT_1} - \frac{1}{k_BT_2}\right]\delta E_1&\\
-\frac{1}{2}\left[\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right) + \frac{1}{k_BT_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)\right]\delta E_1^2 + \cdots&
\end{aligned}
$$

آنسامبل میکروکانونیک

$$\ln\Omega(E,\bar{E}_1+\delta E_1) = \ln\Omega_1(\bar{E}_1)+\ln\Omega_2(E-\bar{E}_1) + \left[\frac{1}{k_BT_1}-\frac{1}{k_BT_2}\right]\delta E_1 - \frac{1}{2}\left[\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right)+\frac{1}{k_BT_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)\right]\delta E_1^2+\cdots$$

اگر T_1 و T_2 با هم برابر باشند،

$$\ln\Omega(E,\bar{E}_1+\delta E_1) = \ln\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)+ - \frac{1}{2}\left[\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right)+\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_2}\right)\right]\delta E_1^2+\cdots$$

$$\ln\Omega(E,\bar{E}_1+\delta E_1)=\ln\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)+ \\ -\frac{1}{2}\left[\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right)+\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_2}\right)\right]\delta E_1^2+\cdots$$

$$\ln\left[\frac{\Omega(E,\bar{E}_1+\delta E_1)}{\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)}\right]=-\frac{1}{2}\left[\frac{1}{k_BT_1^2}\left(\frac{\partial T_1}{\partial \bar{E}_1}\right)+\frac{1}{k_BT_2^2}\left(\frac{\partial T_2}{\partial \bar{E}_2}\right)\right]\delta E_1^2+\cdots$$

$$T_1=T_2,\qquad \left(\frac{\partial T_1}{\partial \bar{E}_1}\right)=\frac{1}{C_v^{(1)}},\qquad \left(\frac{\partial T_2}{\partial \bar{E}_2}\right)=\frac{1}{C_v^{(2)}}$$

$$\ln\left[\frac{\Omega(E,\bar{E}_1+\delta E_1)}{\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)}\right]=-\frac{1}{2}\frac{1}{k_BT^2}\left[\frac{1}{C_v^{(1)}}+\frac{1}{C_v^{(2)}}\right]\delta E_1^2+\cdots$$

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$$\ln\left[\frac{\Omega(E,\bar{E}_1+\delta E_1)}{\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)}\right] \simeq -\frac{1}{2}\frac{1}{k_B T^2}\left[\frac{1}{C_v^{(1)}}+\frac{1}{C_v^{(2)}}\right]\delta E_1^2$$

$$\frac{\Omega(E,\bar{E}_1+\delta E_1)}{\Omega_1(\bar{E}_1)\Omega_2(E-\bar{E}_1)} \simeq e^{-\frac{1}{2}\frac{1}{k_B T^2}\left[\frac{1}{C_v^{(1)}}+\frac{1}{C_v^{(2)}}\right]\delta E_1^2}$$

$$\alpha = \frac{1}{2}\frac{1}{k_B T^2}\left[\frac{1}{C_v^{(1)}}+\frac{1}{C_v^{(2)}}\right]$$

$$E = \bar{E}_1+\bar{E}_2, \quad \Omega(E,\bar{E}_1+\delta E_1) = \Omega(\bar{E}_1,\bar{E}_2)$$

$$\boxed{\frac{\Omega(\bar{E}_1,\bar{E}_2)}{\Omega_1(\bar{E}_1)\Omega_2(\bar{E}_2)} \simeq e^{-\alpha\delta E_1^2}}$$

آنسامبل میکروکانونیک

با فرض اینکه آنتروپی S و تعداد حالتهای Ω بصورت $S = f(\Omega)$ با هم رابطه داشته باشند. نشان دهید که با توجه به خاصیت جمع‌پذیری S و خاصیت ضرب‌پذیری Ω، نیاز است که تابع $f(\Omega)$ تابع از لگاریتم Ω باشد.

$$S = f(\Omega) \Rightarrow f(\Omega) \propto \ln \Omega$$

$$\Omega = \Omega_1 \Omega_2$$

$$f(\Omega_1 \Omega_2) = f(\Omega_1) + f(\Omega_2)$$

$$\frac{\partial}{\partial \Omega_1} f(\Omega_1 \Omega_2) = \frac{\partial}{\partial \Omega_1} f(\Omega_1) \Rightarrow \Omega_2 f'(\Omega_1 \Omega_2) = f'(\Omega_1) \quad (۱)$$

$$\frac{\partial}{\partial \Omega_2} f(\Omega_1 \Omega_2) = \frac{\partial}{\partial \Omega_2} f(\Omega_2) \Rightarrow \Omega_1 f'(\Omega_1 \Omega_2) = f'(\Omega_2) \quad (۲)$$

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$$\frac{①}{②} : \frac{\Omega_2}{\Omega_1} = \frac{f'(\Omega_1)}{f'(\Omega_2)} \Rightarrow \Omega_1 f'(\Omega_1) = \Omega_2 f'(\Omega_2)$$

اگر $f'(\Omega) \propto \frac{1}{\Omega}$ پس

$$f'(\Omega_1) = \frac{a}{\Omega_1} + b$$

و

$$f'(\Omega_2) = \frac{a}{\Omega_2} + b$$

$$\Omega_1 \left(\frac{a}{\Omega_1} + b \right) = \Omega_2 \left(\frac{a}{\Omega_2} + b \right)$$

نتیجه می شود که

$$a \neq 0 \text{ and } b = 0 \Rightarrow f'(\Omega) = \frac{a}{\Omega}$$

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$$f'(\Omega) = \frac{a}{\Omega}$$

اگر از عبارت بالا برحسب Ω انتگرال گرفته شود، داریم

$$f(\Omega) = a \ln \Omega + c$$

که c یک ثابت می‌باشد. با فرض اینکه $S = f(\Omega)$ می‌توان نتیجه گرفت که

$$a = k_B \text{ and } c \neq 0\cdot$$

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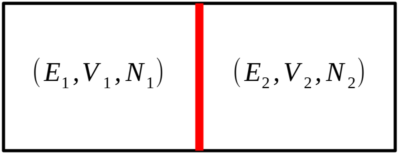
میکروحالت مرکب



$$\Omega(E_1, E_2) = \Omega_1(E_1)\Omega_2(E_2)$$

$$E = E_1 + E_2 \Rightarrow E_2 = E - E_1$$

$$\Omega(E, E_1) = \Omega_1(E_1)\Omega_2(E - E_1)$$

حالت تعادلی سیستم در E_1 وقتی اتفاق می‌افتد که بتواند مقدار میکروحالت مرکب را به حداکثر برساند. یعنی،

$$\left[\frac{\mathrm{d}}{\mathrm{d}E_1}\Omega\right]_{E_1=\bar{E}_1} = 0$$

$$\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1}\Omega_2(E_2) + \Omega_1(E_1)\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_1} = 0$$

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$$\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1}\Omega_2(E_2) + \Omega_1(E_1)\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2}\frac{\mathrm{d}E_2}{\mathrm{d}E_1} = 0$$

$$E_2 = E - E_1$$

$$\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1}\Omega_2(E_2) - \Omega_1(E_1)\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2} = 0$$

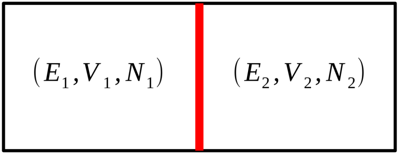
$$\div\Omega_1(E_1)\Omega_2(E_2): \quad \frac{1}{\Omega_1(E_1)}\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1} - \frac{1}{\Omega_2(E_2)}\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2} = 0$$

$$\frac{1}{\Omega_1(E_1)}\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1} = \frac{1}{\Omega_2(E_2)}\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2}$$

$$\frac{1}{\Omega(E)}\frac{\mathrm{d}\Omega(E)}{\mathrm{d}E} = \text{const.} \Rightarrow \frac{1}{\Omega(E)}\frac{\mathrm{d}\Omega(E)}{\mathrm{d}E} = \frac{1}{k_B T} \Rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}E}\ln\Omega(E) = \frac{1}{k_B T}}$$

آنسامبل میکروکانونیک

میکروحالت مرکب



$$\Omega(N_1, N_2) = \Omega_1(N_1)\Omega_2(N_2)$$

$$N = N_1 + N_2 \Rightarrow N_2 = N - N_1$$

$$\Omega(N, N_1) = \Omega_1(N_1)\Omega_2(N - N_1)$$

حالت تعادلی سیستم در N_1 وقتی اتفاق می‌افتد که بتواند مقدار میکروحالت مرکب را به حداکثر برساند. یعنی،

$$\left[\frac{\mathrm{d}}{\mathrm{d}N_1}\Omega\right]_{N_1=\bar{N}_1} = 0$$

$$\frac{\mathrm{d}\Omega_1(N_1)}{\mathrm{d}N_1}\Omega_2(N_2) + \Omega_1(N_1)\frac{\mathrm{d}\Omega_2(N_2)}{\mathrm{d}N_1} = 0$$

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$$\frac{\mathrm{d}\Omega_1(N_1)}{\mathrm{d}N_1}\Omega_2(N_2) + \Omega_1(N_1)\frac{\mathrm{d}\Omega_2(N_2)}{\mathrm{d}N_2}\frac{\mathrm{d}N_2}{\mathrm{d}N_1} = 0$$

$$N_2 = N - N_1$$

$$\frac{\mathrm{d}\Omega_1(N_1)}{\mathrm{d}N_1}\Omega_2(N_2) - \Omega_1(N_1)\frac{\mathrm{d}\Omega_2(N_2)}{\mathrm{d}N_2} = 0$$

$$\div\Omega_1(N_1)\Omega_2(N_2): \quad \frac{1}{\Omega_1(N_1)}\frac{\mathrm{d}\Omega_1(N_1)}{\mathrm{d}N_1} - \frac{1}{\Omega_2(N_2)}\frac{\mathrm{d}\Omega_2(N_2)}{\mathrm{d}N_2} = 0$$

$$\frac{1}{\Omega_1(N_1)}\frac{\mathrm{d}\Omega_1(N_1)}{\mathrm{d}N_1} = \frac{1}{\Omega_2(N_2)}\frac{\mathrm{d}\Omega_2(N_2)}{\mathrm{d}N_2}$$

$$\frac{1}{\Omega(N)}\frac{\mathrm{d}\Omega(N)}{\mathrm{d}N} = \text{const.} \Rightarrow \frac{1}{\Omega(E)}\frac{\mathrm{d}\Omega(N)}{\mathrm{d}N} = -\frac{\mu}{k_B T} \Rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}N}\ln\Omega(N) = -\frac{\mu}{k_B T}}$$

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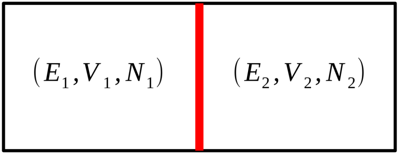
میکروحالت مرکب



$$\Omega(E_1, N_1; E_2, N_2) = \Omega_1(E_1, N_1)\Omega_2(E$$

$$E = E_1 + E_2 \Rightarrow E_2 = E - E_1$$

$$N = N_1 + N_2 \Rightarrow N_2 = N - N_1$$

$$\mathrm{d}\Omega = \left(\frac{\partial}{\partial E_1}\Omega_1(E_1, N_1)\Omega_2(E_2, N_2)\right)\mathrm{d}E_1 + \left(\frac{\partial}{\partial N_1}\Omega_1(E_1, N_1)\Omega_2(E_2, N_2)\right)\mathrm{d}N_1 = 0$$

$$\left(\frac{\partial\Omega_1}{\partial E_1}\Omega_2 + \Omega_1\frac{\partial\Omega_2}{\partial E_1}\right)\mathrm{d}E_1 + \left(\frac{\partial\Omega_1}{\partial N_1}\Omega_2 + \Omega_1\frac{\partial\Omega_2}{\partial N_1}\right)\mathrm{d}N_1 = 0$$

$$\left(\frac{\partial\Omega_1}{\partial E_1}\Omega_2 - \Omega_1\frac{\partial\Omega_2}{\partial E_2}\right)\mathrm{d}E_1 + \left(\frac{\partial\Omega_1}{\partial N_1}\Omega_2 - \Omega_1\frac{\partial\Omega_2}{\partial N_2}\right)\mathrm{d}N_1 = 0$$

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$$\left(\frac{\partial \Omega_1}{\partial E_1}\Omega_2 - \Omega_1\frac{\partial \Omega_2}{\partial E_2}\right)\mathrm{d}E_1 + \left(\frac{\partial \Omega_1}{\partial N_1}\Omega_2 - \Omega_1\frac{\partial \Omega_2}{\partial N_2}\right)\mathrm{d}N_1 = 0$$

$$\left(\frac{1}{\Omega_1}\frac{\partial \Omega_1}{\partial E_1} - \frac{1}{\Omega_2}\frac{\partial \Omega_2}{\partial E_2}\right)\mathrm{d}E_1 + \left(\frac{1}{\Omega_1}\frac{\partial \Omega_1}{\partial N_1} - \frac{1}{\Omega_2}\frac{\partial \Omega_2}{\partial N_2}\right)\mathrm{d}N_1 = 0$$

$$\frac{1}{\Omega_1}\frac{\partial \Omega_1}{\partial E_1} = \frac{1}{k_B T_1}, \qquad \frac{1}{\Omega_2}\frac{\partial \Omega_2}{\partial E_2} = \frac{1}{k_B T_2}$$

$$\frac{1}{\Omega_1}\frac{\partial \Omega_1}{\partial N_1} = -\frac{\mu_1}{k_B T_1}, \qquad \frac{1}{\Omega_2}\frac{\partial \Omega_2}{\partial N_2} = \frac{\mu_2}{k_B T_2}$$

$$\left(\frac{1}{k_B T_1} - \frac{1}{k_B T_2}\right)\mathrm{d}E_1 + \left(-\frac{\mu_1}{k_B T_1} + \frac{\mu_2}{k_B T_2}\right)\mathrm{d}N_1 = 0$$

$$\left(\frac{1}{k_B T_1} - \frac{1}{k_B T_2}\right)\mathrm{d}E_1 = \left(\frac{\mu_1}{k_B T_1} - \frac{\mu_2}{k_B T_2}\right)\mathrm{d}N_1$$

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$$\left(\frac{T_2 - T_1}{k_B T_1 T_2}\right) \mathrm{d}E_1 = \left(\frac{\mu_1 T_2 - \mu_2 T_1}{k_B T_1 T_2}\right) \mathrm{d}N_1$$

$$\left(\frac{T_2 - T_1}{\color{red}{k_B T_1 T_2}}\right) \mathrm{d}E_1 = \left(\frac{\mu_1 T_2 - \mu_2 T_1}{\color{red}{k_B T_1 T_2}}\right) \mathrm{d}N_1$$

$$\boxed{\frac{\mathrm{d}E_1}{\mathrm{d}N_1} = \frac{\mu_1 T_2 - \mu_2 T_1}{T_2 - T_1}}$$

آنسامبل میکروکانونیک

حجم کره n بعدی

$$V_n = \int\limits_{x_1^2+\cdots+x_n^2 \le R^2} \mathrm{d}x_1 \cdots \int \mathrm{d}x_n \propto R^n$$

مثال دوبعدی

$$\int\limits_{x_1^2+x_2^2 \le R^2} \mathrm{d}x_1 \int \mathrm{d}x_2 = \pi R^2$$

مثال سه‌بعدی

$$\int\limits_{x_1^2+x_2^2+x_3^2 \le R^2} \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3 = \frac{4\pi}{3} R^2$$

$$V_n = \int\limits_{x_1^2+\cdots+x_n^2 \le R^2} \mathrm{d}x_1 \cdots \int \mathrm{d}x_n = C_n R^n, \quad C_n = ?$$

آنسامبل میکروکانونیک

حجم کره n بعدی

$$V_n = \underset{x_1^2+\cdots+x_n^2 \le R^2}{\int \mathrm{d}x_1 \cdots \int \mathrm{d}x_n} = C_n R^n, \quad C_n = ?$$

$$\mathrm{d}V_n = n C_n R^{n-1} \mathrm{d}R$$

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n e^{-(x_1^2+\cdots+x_n^2)} = \left(\int_{-\infty}^{\infty} e^{-x_1^2} \mathrm{d}x_1\right) \cdots \left(\int_{-\infty}^{\infty} e^{-x_n^2} \mathrm{d}x_n\right)$$

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n e^{-(x_1^2+\cdots+x_n^2)} = \pi^{n/2} \quad \text{(I)}$$

برای یک کره n بعدی می‌توان المان $\mathrm{d}x_1 \cdots \mathrm{d}x_n$ را با المان $n C_n R^{n-1} \mathrm{d}R$ عوض کرد،

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n e^{-(x_1^2+\cdots+x_n^2)} = n C_n \int_0^{\infty} e^{-R^2} R^{n-1} \mathrm{d}R \quad \text{(II)}$$

آنسامبل میکروکانونیک

حجم کره n بعدی

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n e^{-(x_1^2+\cdots+x_n^2)} = \pi^{n/2} \qquad \text{(I)}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n e^{-(x_1^2+\cdots+x_n^2)} = nC_n \int_0^{\infty} e^{-R^2} R^{n-1} \mathrm{d}R \qquad \text{(II)}$$

بنابراین

$$\pi^{n/2} = nC_n \int_0^{\infty} e^{-R^2} R^{n-1} \mathrm{d}R$$

$$\int_0^{\infty} e^{-R^2} R^{n-1} \mathrm{d}R, \quad R^2 = y, \quad 2R\mathrm{d}R = \mathrm{d}y$$

$$\int_0^{\infty} e^{-R^2} R^{n-1} \mathrm{d}R = \frac{1}{2} \int_0^{\infty} e^{-y} y^{\frac{n}{2}-1} \mathrm{d}y = \frac{1}{2}\Gamma(\frac{n}{2} - 1)$$

آنسامبل میکروکانونیک

حجم کره n بعدی

$$\pi^{n/2} = nC_n \int_0^\infty e^{-R^2} R^{n-1} \mathrm{d}R$$

$$\int_0^\infty e^{-R^2} R^{n-1} \mathrm{d}R = \frac{1}{2}\Gamma(\frac{n}{2} - 1)$$

$$\pi^{n/2} = C_n \frac{n}{2}\Gamma(\frac{n}{2} - 1) \Rightarrow \pi^{n/2} = C_n \Gamma(\frac{n}{2})$$

$$C_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

$$\boxed{V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} R^n, \quad \mathrm{d}V_n = n\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} R^{n-1} \mathrm{d}R}$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$\mathcal{H} = \sum_{i}^{N} \frac{\vec{p}_i \cdot \vec{p}_i}{2m} = \sum_{i}^{3N} \frac{p_i^2}{2m}$$

حجم نهایی فضای فاز

$$\Phi(E,V,N) = \frac{1}{N!}\frac{1}{h^{3N}} \int_{\mathcal{H}\leq E} \prod_{i}^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \underbrace{\int \mathrm{d}p_1 \cdots \int \mathrm{d}p_{3N}}_{\mathcal{H}(p_1,\cdots,p_{3N})\leq E}$$

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \underbrace{\int \mathrm{d}p_1 \cdots \int \mathrm{d}p_{3N}}_{\frac{p_1^2}{2m}+\cdots+\frac{p_{3N}^2}{2m}\leq E}$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \int\limits_{p_1^2+\cdots+p_{3N}^2 \le 2mE} \mathrm{d}p_1 \cdots \int \mathrm{d}p_{3N}$$

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \int\limits_{p_1^2+\cdots+p_{3N}^2 \le (\sqrt{2mE})^2} \mathrm{d}p_1 \cdots \int \mathrm{d}p_{3N}$$

$$\int\limits_{x_1^2+\cdots+x_n^2 \le R^2} \mathrm{d}x_1 \cdots \int \mathrm{d}x_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} R^n$$

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \int\limits_{p_1^2+\cdots+p_{3N}^2 \le (\sqrt{2mE})^2} \mathrm{d}p_1 \cdots \int \mathrm{d}p_{3N} = V^N \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})} (2mE)^{3N/2}$$

$$\boxed{\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})} (2mE)^{3N/2}}$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$\Phi(E,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})}(2mE)^{3N/2}$$

تعداد حالتها بازای $E \leq \mathcal{H} \leq E + \delta E$

$$\Omega(E,V,N) = \frac{\partial \Phi}{\partial E}\delta E, \qquad |\frac{\delta E}{E}| \ll 1$$

$$\boxed{\Omega(E,V,N) = \frac{1}{N!}\frac{3N}{2}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})}(2mE)^{3N/2-1}\delta E}$$

آنتروپی

$$\boxed{S = k_B \ln \Omega = k_B \ln \left[\frac{1}{N!}\frac{3N}{2}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})}(2mE)^{3N/2-1}\delta E\right]}$$

آنسامبل میکروکانونیک
گاز ایده‌آل تمییزناپذیر

$$S = k_B \ln \left[\frac{1}{N!} \frac{3N}{2} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2})} (2mE)^{3N/2} \left(\frac{\delta E}{E} \right) \right]$$

برای Nهای بزرگ

$$\Gamma(\frac{3N}{2}) = (\frac{3N}{2} - 1)! \approx (\frac{3N}{2})!$$

$$S = k_B \ln \left[\frac{1}{N!} \frac{3N}{2} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{(\frac{3N}{2})!} (2mE)^{3N/2} \left(\frac{\delta E}{E} \right) \right]$$

$$S = N k_B \ln \left[\frac{V}{h^3} (2m\pi E)^{3/2} \right] + k_B \ln \left(\frac{\delta E}{E} \right) + k_B \ln \left(\frac{3N}{2} \right) - k_B \ln (\frac{3N}{2})! - k_B \ln N!$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$S = Nk_B \ln\left[\frac{V}{h^3}(2m\pi E)^{3/2}\right] + k_B \ln\left(\frac{\delta E}{E}\right) + k_B \ln\left(\frac{3N}{2}\right) - k_B \ln(\frac{3N}{2})! - k_B \ln N!$$

برای Nهای بزرگ

$$\left(\frac{3N}{2}\right)! = \left(\frac{3N}{2}\right)\ln\left(\frac{3N}{2}\right) - \left(\frac{3N}{2}\right), \quad \ln N! = N\ln N - N$$

و اینکه

$$|\frac{\delta E}{E}| \ll 1 \Rightarrow \ln\left(\frac{\delta E}{E}\right) \to 0, \quad \ln\left(\frac{3N}{2}\right) \to 0$$

$$S = Nk_B \ln\left[\frac{V}{h^3}(2m\pi E)^{3/2}\right] - k_B\left(\frac{3N}{2}\right)\ln\left(\frac{3N}{2}\right) + k_B\left(\frac{3N}{2}\right) - k_B N\ln N + k_B N$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$S = Nk_B \ln\left[\frac{V}{h^3}(2m\pi E)^{3/2}\right] - Nk_B \ln\left(\frac{3N}{2}\right)^{3/2} + k_B\left(\frac{5N}{2}\right) - k_B N \ln N$$

$$S = Nk_B \ln\left[\frac{V}{Nh^3}\left(\frac{4m\pi E}{3N}\right)^{3/2}\right] + k_B\left(\frac{5N}{2}\right)$$

$$\boxed{S = Nk_B\left(\ln\left[\frac{V}{N}\left(\frac{4m\pi E}{3Nh^2}\right)^{3/2}\right] + \frac{5}{2}\right)}, \qquad \text{Sackur–Tetrode}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{3N}{2}\frac{E^{3N/2-1}}{E^{3N/2}} = \frac{3N}{2}\frac{1}{E} \Rightarrow E = \frac{3}{2}Nk_B T$$

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{N,E} = N\frac{V^{N-1}}{V^N} = N\frac{1}{V} \Rightarrow pV = Nk_B T$$

آنسامبل میکروکانونیک

گاز ایده‌آل تمییزناپذیر

$$E = \frac{3}{2} N k_B T$$

$$pV = N k_B T$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3}{2} N k_B$$

$$C_p = C_V + \left[\left(\frac{\partial E}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p = \frac{3}{2} N k_B + (0+p)\frac{N k_B}{p}$$

$$C_p = \frac{3}{2} N k_B + N k_B = \frac{5}{2} N k_B$$

آنسامبل میکروکانونیک

نوسانگر ساده تمییزپذیر

$$\mathcal{H} = \sum_{i}^{3N} \frac{p_i^2}{2m} + \frac{1}{2}kq_i^2$$

حجم نهایی فضای فاز

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int_{\mathcal{H}\leq E} \prod_{i}^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int_{\mathcal{H}(p_1,\cdots,p_{3N},q_1,\cdots,q_{3N})\leq E} \prod_{i}^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int_{\frac{p_1^2}{2m}+\cdots+\frac{p_{3N}^2}{2m}+\frac{1}{2}kq_1^2+\cdots+\frac{1}{2}kq_{3N}^2\leq E} \prod_{i}^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

آنسامبل میکروکانونیک
نوسانگر ساده تمییزپذیر

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int\limits_{\frac{p_1^2}{2m}+\cdots+\frac{p_{3N}^2}{2m}+\frac{1}{2}kq_1^2+\cdots+\frac{1}{2}kq_{3N}^2 \leq E} \prod_i^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int\limits_{\left(\frac{p_1}{\sqrt{2mE}}\right)^2+\cdots+\left(\frac{p_{3N}}{\sqrt{2mE}}\right)^2+\left(\frac{q_1}{\sqrt{\frac{2E}{k}}}\right)^2+\cdots+\left(\frac{q_{3N}}{\sqrt{\frac{2E}{k}}}\right)^2 \leq 1} \prod_i^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$y_i = \frac{p_i}{\sqrt{2mE}} \Rightarrow \mathrm{d}p_i = \sqrt{2mE}\,\mathrm{d}y_i$$

$$x_i = \frac{q_i}{\sqrt{\frac{2E}{k}}} \Rightarrow \mathrm{d}q_i = \sqrt{\frac{2E}{k}}\,\mathrm{d}x_i$$

آنسامبل میکروکانونیک

نوسانگر ساده تمییزپذیر

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int\limits_{\frac{p_1^2}{2m}+\cdots+\frac{p_{3N}^2}{2m}+\frac{1}{2}kq_1^2+\cdots+\frac{1}{2}kq_{3N}^2\leq E} \prod_i^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}} \int\limits_{\left(\frac{p_1}{\sqrt{2mE}}\right)^2+\cdots+\left(\frac{p_{3N}}{\sqrt{2mE}}\right)^2+\left(\frac{q_1}{\sqrt{\frac{2E}{k}}}\right)^2+\cdots+\left(\frac{q_{3N}}{\sqrt{\frac{2E}{k}}}\right)^2\leq 1} \prod_i^{3N} \mathrm{d}q_i \mathrm{d}p_i$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}} (2mE)^{3N/2} \left(\frac{2E}{k}\right)^{3N/2} \int\limits_{y_1^2+\cdots+y_{3N}^2+x_1^2+\cdots+x_{3N}^2\leq 1} \prod_i^{3N} \mathrm{d}x_i \mathrm{d}y_i$$

آنسامبل میکروکانونیک

نوسانگر ساده تمییزپذیر

$$\Phi(E,V,N) = \frac{1}{h^{3N}}(2mE)^{3N/2}\left(\frac{2E}{k}\right)^{3N/2} \int\limits_{y_1^2+\cdots+y_{3N}^2+x_1^2+\cdots+x_{3N}^2\leq 1} \prod_i^{3N} \mathrm{d}x_i \mathrm{d}y_i$$

$$\int\limits_{y_1^2+\cdots+y_{3N}^2+x_1^2+\cdots+x_{3N}^2\leq 1} \prod_i^{3N} \mathrm{d}x_i \mathrm{d}y_i = \frac{\pi^{3N}}{(3N)!}$$

$$\Phi(E,V,N) = \frac{1}{h^{3N}}(2mE)^{3N/2}\left(\frac{2E}{k}\right)^{3N/2} \frac{\pi^{3N}}{(3N)!}$$

$$\Phi(E,V,N) = \frac{1}{(3N)!}\left(\sqrt{\frac{m}{k}}\right)^{3N} \frac{(2\pi)^{3N}}{h^{3N}} E^{3N}$$

آنسامبل میکروکانونیک

نوسانگر ساده تمییزپذیر

$$\Phi(E,V,N) = \frac{1}{(3N)!}\left(\sqrt{\frac{m}{k}}\right)^{3N} \frac{(2\pi)^{3N}}{h^{3N}} E^{3N}$$

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \hbar = \frac{h}{2\pi}$$

$$\Phi(E,V,N) = \frac{1}{(3N)!}\left(\frac{E}{\hbar\omega_0}\right)^{3N}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

تعداد حالتها بازای $E \leq \mathcal{H} \leq E + \delta E$

$$\Omega(E,V,N) = \frac{\partial \Phi}{\partial E}\delta E, \quad |\frac{\delta E}{E}| \ll 1$$

$$\Omega(E,V,N) = \frac{3N}{(3N)!}\left(\frac{E}{\hbar\omega_0}\right)^{3N} \frac{\delta E}{E}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

آنسامبل میکروکانونیک

نوسانگر ساده تمیزپذیر

$$\Omega(E,V,N) = \frac{3N}{(3N)!}\left(\frac{E}{\hbar\omega_0}\right)^{3N}\frac{\delta E}{E}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$S = k_B \ln \Omega = k_B \ln\left[\frac{3N}{(3N)!}\frac{E^{3N}}{(\hbar\omega_0)^{3N}}\frac{\delta E}{E}\right]$$

$$S = 3Nk_B \ln\left[\frac{E}{\hbar\omega_0}\right] + k_B \ln(3N) + k_B \ln\left(\frac{\delta E}{E}\right) - 3Nk_B \ln(3N) + 3Nk_B$$

و اینکه

$$|\frac{\delta E}{E}| \ll 1 \Rightarrow \ln\left(\frac{\delta E}{E}\right) \to 0, \quad \ln(3N) \to 0$$

$$S = 3Nk_B \ln\left[\frac{E}{3N\hbar\omega_0}\right] + 3Nk_B$$

آنسامبل میکروکانونیک

نوسانگر ساده تمییزپذیر

$$S = 3Nk_B \ln\left[\frac{E}{3N\hbar\omega_0}\right] + 3Nk_B$$

$$S = 3Nk_B \left(\ln\left[\frac{E}{3N\hbar\omega_0}\right] + 1\right)$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} = 3Nk_B\frac{1}{E} \Rightarrow E = 3Nk_BT$$

$$S = 3Nk_B \left(\ln\left[\frac{k_BT}{\hbar\omega_0}\right] + 1\right)$$

$$\frac{p}{T} = \frac{\partial S}{\partial V} = 0 \Rightarrow p = 0$$

$$E = 3Nk_BT$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3Nk_B$$

$$C_p = C_V + \left[\left(\frac{\partial E}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p = 3Nk_B + 0 = 3Nk_B$$

$$-\frac{\mu}{T} = \frac{\partial S}{\partial N} = 3k_B\left(\ln\left[\frac{E}{3N\hbar\omega_0}\right] + 1\right) - 3k_B$$

$$-\frac{\mu}{T} = 3k_B \ln\left[\frac{E}{3N\hbar\omega_0}\right]$$

$$\mu = -3k_BT\ln\left[\frac{E}{3N\hbar\omega_0}\right] = -3k_BT\ln\left[\frac{k_BT}{\hbar\omega_0}\right]$$